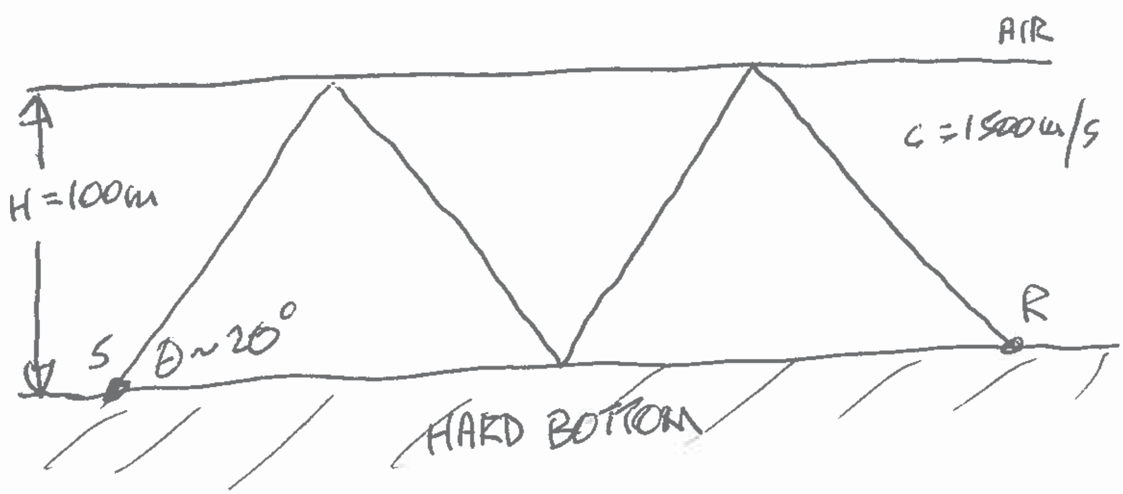
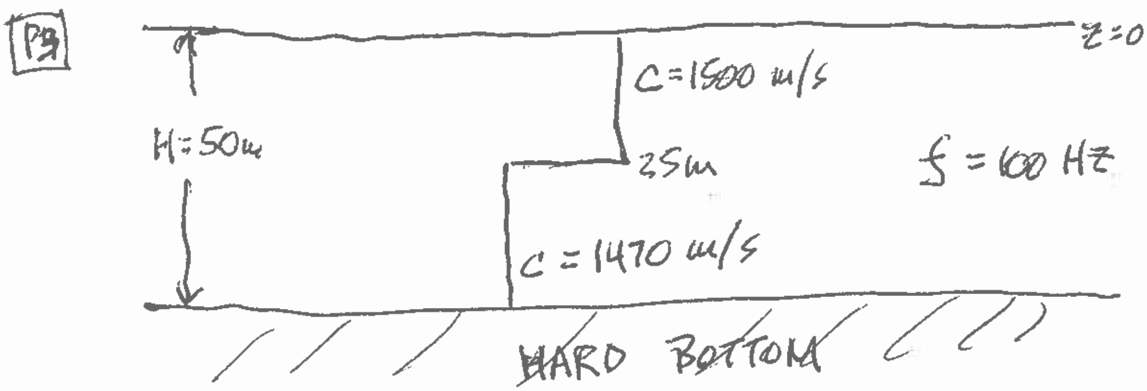
Problem Set #4. (Due two weeks from assignment date).

1) Mode filtering. Consider a 30m deep, isovelocity water column (1500 m/s) hard bottom waveguide with a 100 Hz source transmitting mid-column at R=0 and a five element vertical array receiver array at 10 km away.
   a) Create a picture of the sound intensity (dB) in the region of the receiver (+/- 1km).
   b) Filter, using five evenly spaced elements, the normal modes at R=10 km, i.e. estimate the mode amplitudes a_m . Are the modes perfectly filtered, or is there "leakage" to other modes?
   c) If one takes the vertical array to span the bottom half of the water column and have infinitely many elements (i.e. an integral approximation), how well do you now filter the modes? Show the mode amplitudes and leakage. This is actually not a strictly academic exercise, as experimentally you don't want an array up near the surface due to wave effects and possible ship strikes.
2) Ray/mode picture. The ray/mode picture correspondence can be easily demonstrated on a computer by creating "fuzzy rays" out of normal modes. Let's give it a try! First, put a source and receiver on the bottom of a 100m deep, hard bottom, isovelocity waveguide (usual 1500 m/s water column speed) and consider a path as shown below in Figure P2.
   a) For 100, 500, and 1000 Hz, consider the modes near a grazing angle of around 20 degrees (doesn't have to be exactly, just close). Find the source/receiver distance (mode cycle distance) for each frequency.
   b) Find the modes that are around 20 degrees that might contribute to the coherent sum, i.e. close to 2π phase differences along the path.
   c) Add up the complex pressures of these neighbor modes and then look at $|p(r,z)|$ (or intensity, whichever shows more clearly) in a 2-D black and white plot. You should see the intensity track the equivalent ray around 20 degrees, and get thinner as the frequency increases.



3) Perturbation theory. In HWK #3, Problem #4, you generated the 100 Hz mode eigenvalues for a hard bottom waveguide with a two-layer water column profile, shown in Figure P3. That was an "exact solution", i.e. you run the mode code and grab the answer. Now suppose you're traveling somewhere, and you just have a hand calculator, but not a computer. (More and more unlikely these days, but remotely possible!) Since you know about perturbation theory and you know the solution to the hard bottom waveguide problem, you can actually obtain (approximate) solutions for the modal eigenvalues with a calculator. Let's do so!

a) Generate the 100 Hz mode eigenvalues for the "average water column" background profile. (Trivial – is just the c=1485 m/s hard bottom.)



b) Now generate the mode eigenvalues for the two layer waveguide. Compare your results to both the background profile and to the exact solutions from your previous homework. Do you see any trends?

c) This is one I haven't tried, but I think it should work. Gotta give it a shot! Here's the idea. You can also get the mode eigenvectors from perturbation theory, but even in first order, they're kinda ugly expressions, and certainly not "calculator friendly." However, given that we know that the hard bottom modes are sinusoids, and we have the perturbed eigenvalues, could we use our "shooting technique" and just propagate $\sin(\gamma_m z)$ down from the surface initially to get (approximately) the perturbed mode?! Please try it! Can you see where this fails? (If you can check against exact and/or perturbed expression, is worth extra credit.)

4) Perturbation theory. Here's a more general system for perturbation theory – a simple pendulum! In general, the equation for a simple pendulum's motion is a nonlinear one, but for small angles, it is simple and linear. This is explained in the handout notes.

a) Can the perturbation theory we worked out in class describe the change in the period due to large angle effects? (The handout shows this with a trial solution, but we want to use perturbation theory.) Please show this, and describe how the errors increase with increasing angle.

b) For extra credit, how could I very simply make a pendulum that kept a constant period?!? (Hint: this is a tautochrone problem.)

"SIMPLE (?!)" PENDULUM

$$\frac{d^2}{dt^2}\theta + \frac{g}{l}\theta = 0$$ KNOW (UNPERTURBED)

$$\frac{d^2}{dt^2}\theta + \frac{g}{l}\sin\theta = 0$$ WANT (PERTURBED)

$$\frac{d^2}{dt^2}\theta + \frac{g}{l}\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots\right)$$ EXPANSION OF SIN θ

Also, $\theta^{\text{UNPERT.}} = \theta_0 \cos(\omega_0 t + \phi_0)$ (with $\phi_0 \to 0$)

$$\omega_0 = \sqrt{\frac{g}{l}} \quad \text{and} \quad T_0 = \frac{2\pi}{\omega_0} = 2\pi\sqrt{\frac{l}{g}}$$

2.682 Acoustical Oceanography
Spring 2012