6B.2 Eccentric-Disk Rheometer Flow [TWL]

a. $\nabla\underline{v} = \begin{pmatrix} 0 & W & 0 \\ -W & 0 & 0 \\ AW & 0 & 0 \end{pmatrix}$; $(\nabla\underline{v})^\dagger = \begin{pmatrix} 0 & -W & AW \\ W & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

$$\underline{\underline{\gamma}}_{(1)} = AW\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix};\ \{\underline{\underline{\gamma}}_{(1)}\cdot\underline{\underline{\gamma}}_{(1)}\} = (AW)^2\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\underline{\underline{\gamma}}_{(2)} = -\{(\nabla\underline{v})^\dagger\cdot\underline{\underline{\gamma}}_{(1)} + \underline{\underline{\gamma}}_{(1)}\cdot(\nabla\underline{v})\}$$

$$= -\begin{pmatrix} 0 & -W & AW \\ W & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}AW - \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & W & 0 \\ -W & 0 & 0 \\ AW & 0 & 0 \end{pmatrix}AW$$

$$= -\begin{pmatrix} 2AW & 0 & 0 \\ 0 & 0 & W \\ 0 & W & 0 \end{pmatrix}AW$$

$$\underline{\underline{\gamma}}_{(3)} = -\{(\nabla\underline{v})^\dagger\cdot\underline{\underline{\gamma}}_{(2)} + \underline{\underline{\gamma}}_{(2)}\cdot\nabla\underline{v}\}$$

$$= \begin{pmatrix} 0 & -W & AW \\ W & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 2A & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}AW^2 + \begin{pmatrix} 2A & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & W & 0 \\ -W & 0 & 0 \\ AW & 0 & 0 \end{pmatrix}AW$$

$$= AW^3\begin{pmatrix} 0 & 3A & -1 \\ 3A & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}$$

$$\{\underline{\underline{\gamma}}_{(1)}\cdot\underline{\underline{\gamma}}_{(2)} + \underline{\underline{\gamma}}_{(2)}\cdot\underline{\underline{\gamma}}_{(1)}\} = -A^2W^3\begin{pmatrix} 0 & 1 & 2A \\ 1 & 0 & 0 \\ 2A & 0 & 0 \end{pmatrix}$$

6B.2 (cont'd)

b. From Eq. 6.2-1 we get

$$\tau_{xz} = -\left[b_1 AW + b_2(0) + b_{11}(0) + b_3(-AW^3) + b_{12}(-2A^3W^3) + b_{1:11}(2A^3W^3) \right]$$

$$\tau_{yz} = -\left[b_1(0) + b_2(-AW^2) + b_{11}(0) + b_3(0) + b_{12}(0) + b_{1:11}(0) \right] = b_2 AW^2$$

c. $$\lim_{A\to 0}\left(-\frac{\tau_{xz}}{AW}\right) = \lim_{A\to 0}\left(b_1 - b_3W^2 - 2b_{12}A^2W^2 + 2b_{1:11}A^2W^2\right) = b_1 - b_3W^2$$

$$\lim_{A\to 0}\left(-\frac{\tau_{yz}}{AW}\right) = -b_2W$$

6B.3 Complex Viscosity for Third-Order Fluid [JDS]

a) Small Ampl. Osc. Shear:

$$\underline{\underline{\gamma}}_{(1)} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \dot{\gamma}^{\circ} \operatorname{Re}\{e^{i\omega t}\}$$

since $\dot{\gamma}^{\circ} \ll 1$, keep terms 1st order in $\dot{\gamma}^{\circ}$:

$$\underline{\underline{\gamma}}_{(2)} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \dot{\gamma}^{\circ} \operatorname{Re}\{i\omega e^{i\omega t}\} + \ldots$$

$$\underline{\underline{\gamma}}_{(3)} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \dot{\gamma}^{\circ} \operatorname{Re}\{-\omega^2 e^{i\omega t}\} + \ldots$$

$\underline{\underline{\gamma}}_{(1)} \cdot \underline{\underline{\gamma}}_{(1)}$, $\underline{\underline{\gamma}}_{(1)} \cdot \underline{\underline{\gamma}}_{(2)}$ contain terms 2d order in $\dot{\gamma}^{\circ}$

3d Order fluid:

$$\underline{\underline{\tau}} = -\left[b_1 \underline{\underline{\gamma}}_{(1)} + b_2 \underline{\underline{\gamma}}_{(2)} + b_3 \underline{\underline{\gamma}}_{(3)} + \ldots\right]$$

$$\tau_{yx} = \dot{\gamma}^{\circ} \operatorname{Re}\{\eta^* e^{i\omega t}\}$$

$$\therefore \quad \eta^* = b_1 + b_2 \omega i - b_3 \omega^2$$

b) $\eta' = \mathrm{Re}\{\eta^*\} = b_1 - b_3\omega^2$

$\eta'' = -\mathrm{Im}\{\eta^*\} = -b_2\omega$

Fig. 3.4-4 : Predicts correct 1st Order correction to η' from Ⓝ

Fig 3.4-5 : Predicts correct 1st Order correction to η''/ω

Fig 3.4-6 : Same as above for η' & η''

c) $\lim_{\omega \to 0} \frac{\eta''/\omega}{\eta'} = -\frac{b_2}{b_1}$

From (6.2-5,6): $\lim_{\dot{\gamma} \to 0} \frac{\Psi_1}{2\eta} = -\frac{2b_2}{2b_1} = -\frac{b_2}{b_1}$

d) No.

68.3 a) Small Amplitude Oscillatory Shear Flow : $\dot{\gamma}^\circ << 1$, where

$$\underline{\underline{\gamma}}_{(1)} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \dot{\gamma}^\circ \mathrm{Re}\{e^{i\omega t}\}$$, keep terms only to order $\dot{\gamma}^\circ$ (others are too small.

$$\underline{\underline{\gamma}}_{(2)} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \dot{\gamma}^\circ \mathrm{Re}\{i\omega e^{i\omega t}\} + \text{terms } 2^d \text{order in } \dot{\gamma}^\circ$$

$$\underline{\underline{\gamma}}_{(3)} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \dot{\gamma}^\circ \mathrm{Re}\{-\omega^2 e^{i\omega t}\} + \ldots$$

Note that $\underline{\underline{\gamma}}_{(1)} \cdot \underline{\underline{\gamma}}_{(1)}$, $\underline{\underline{\gamma}}_{(1)} \cdot \underline{\underline{\gamma}}_{(2)}$, contain terms only 2^d or higher order $\dot{\gamma}^\circ$

$$\underline{\underline{\tau}} = -\left[b_1 \underline{\underline{\gamma}}_{(1)} + b_2 \underline{\underline{\gamma}}_{(2)} + b_3 \underline{\underline{\gamma}}_{(3)} + \ldots\right]$$; 3^d order fluid, 1^{st} order in $\dot{\gamma}^\circ$

$$\tau_{yx} = \dot{\gamma}^\circ \mathrm{Re}\{\eta^* e^{i\omega t}\} \longrightarrow \eta^* = b_1 + b_2 \omega i - b_3 \omega^2$$

b) $\eta' = \mathrm{Re}\{\eta^*\} = b_1 - b_3\omega^2$; $\eta'' = -\mathrm{Im}\{\eta^*\} = -b_2\omega$

Fig 3.4-4 → Predicts the correct 1^{st} order correction for η' from const. (N) predi

Fig 3.4-5 → " " " " " " η''/ω, that is a constant

Fig 3.4-6 → Same as above for η' & η''

c) $\lim_{\omega \to 0} \dfrac{\eta''/\omega}{\eta'} = -b_2/b_1$; From Eqs. (6.2-5,6) : $\lim_{\dot{\gamma} \to 0} \dfrac{\Psi_1}{2\eta} = \dfrac{-2b_2}{2b_1} = -$

d) No, $\Psi_1 \neq 0$ only for non-linear VE.

6B.12 The Second-Order Fluid and the "Turntable Experiment" [RBB]

a. $$\underline{\underline{\gamma}}_{(1)}(t) = \begin{pmatrix} -\sin 2Wt & \cos 2Wt & 0 \\ \cos 2Wt & \sin 2Wt & 0 \\ 0 & 0 & 0 \end{pmatrix}\dot{\gamma}$$

$$\overset{t=0}{\Longrightarrow} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\dot{\gamma}$$

$$\{\underline{\underline{\gamma}}_{(1)}\cdot\underline{\underline{\gamma}}_{(1)}\} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}\dot{\gamma}^2$$ ← No "t" appears here.

b. $$\frac{\partial \underline{\underline{\gamma}}_{(1)}}{\partial t} = \begin{pmatrix} -\cos 2Wt & -\sin 2Wt & 0 \\ -\sin 2Wt & \cos 2Wt & 0 \\ 0 & 0 & 0 \end{pmatrix} 2W\dot{\gamma}$$

$$\overset{t=0}{\Longrightarrow} \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} 2W\dot{\gamma}$$

$$\nabla\underline{v} = \begin{pmatrix} -\dot{\gamma}\sin Wt\cos Wt & -\dot{\gamma}\sin^2 Wt + W & 0 \\ \dot{\gamma}\cos^2 Wt - W & \dot{\gamma}\sin Wt\cos Wt & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\underline{\underline{\omega}} = \nabla\underline{v} - (\nabla\underline{v})^\dagger = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}(\dot{\gamma} - 2W)$$

c. $$\underline{\underline{\gamma}}_{(2)} = \frac{D\underline{\underline{\gamma}}_{(1)}}{Dt} - \{(\nabla\underline{v})^\dagger\cdot\underline{\underline{\gamma}}_{(1)} + \underline{\underline{\gamma}}_{(1)}\cdot\nabla\underline{v}\}$$

Since $\nabla\underline{v} = \frac{1}{2}(\underline{\underline{\gamma}}_{(1)} + \underline{\underline{\omega}})$ and $(\nabla\underline{v})^\dagger = \frac{1}{2}(\underline{\underline{\gamma}}_{(1)} - \underline{\underline{\omega}})$ we have:

$$\underline{\underline{\gamma}}_{(2)} = \frac{D\underline{\underline{\gamma}}_{(1)}}{Dt} - \{\underline{\underline{\gamma}}_{(1)}\cdot\underline{\underline{\gamma}}_{(1)}\} + \frac{1}{2}\{\underline{\underline{\omega}}\cdot\underline{\underline{\gamma}}_{(1)} - \underline{\underline{\gamma}}_{(1)}\cdot\underline{\underline{\omega}}\}$$

d. Note that for the flow under consideration $\{\underline{v} \cdot \nabla \underline{\underline{\gamma}}_{(1)}\} = 0$. Then:

$$\underline{\underline{\gamma}}_{(2)}\Big|_{t=0} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} 2W\dot{\gamma} - \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \dot{\gamma}^2$$

$$+ \frac{1}{2}\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \dot{\gamma}(\dot{\gamma} - 2W)$$

$$- \frac{1}{2}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \dot{\gamma}(\dot{\gamma} - 2W)$$

$$= \cancel{\begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} 2W\dot{\gamma}} - \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \dot{\gamma}^2$$

$\boxed{\text{from } \dfrac{\partial \underline{\underline{\gamma}}_{(1)}}{\partial t}}$ $\boxed{\text{from } \underline{\underline{\gamma}}_{(1)} \cdot \underline{\underline{\gamma}}_{(1)}}$

$$+ \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \dot{\gamma}(\dot{\gamma} - \cancel{2W})$$

$\boxed{\text{from } \underline{\underline{\omega}}\text{-terms}}$

$$= -\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} 2\dot{\gamma}^2$$

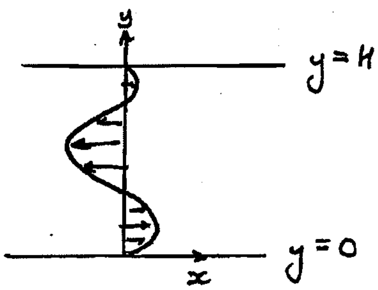
e.
$$\begin{pmatrix} \tau_{xx} & \tau_{xy} & 0 \\ \tau_{yx} & \tau_{yy} & 0 \\ 0 & 0 & \tau_{zz} \end{pmatrix} = -b_1 \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \dot{\gamma}$$

$$+ b_2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} 2\dot{\gamma}^2 - b_{11} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \dot{\gamma}^2$$

This is the same as Eq. 6.2-4; there is no dependence on W.

f. If $\underline{\underline{\gamma}}_{(2)}$ is replaced by $\partial \underline{\underline{\gamma}}_{(1)}/\partial t$ in the above, there would be no $\underline{\underline{\omega}}$-terms, and there would be a W-dependence!

5.C.1 STABILITY OF SECOND ORDER FLUIDS [GHM]



B.C. $\quad v_x(0,t) = v_x(H,t) = 0$

I.C. $\quad v_x(y,t) = U(y)$

There is no modified pressure gradient and the flow is RECTILINEAR, $v_y = v_z = 0$

a) The Velocity gradient tensor is $\quad \underline{\nabla v} = \dot\gamma \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ where $\dot\gamma = \dfrac{\partial v_x(y,t)}{\partial y}$

The Kinematic Tensors required for second order Fluid are

$$\underline{\underline{\gamma}}_{(1)} = \dot\gamma \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \underline{\underline{\gamma}}_{(1)} : \underline{\underline{\gamma}}_{(1)} = 2\dot\gamma^2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\underline{\underline{\gamma}}_{(2)} = \frac{\partial \dot\gamma}{\partial t} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} - 2\dot\gamma^2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Hence

$$\underline{\underline{\tau}} = -b_1 \left[\dot\gamma \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \frac{b_2}{b_1} \begin{pmatrix} -2\dot\gamma^2 & \frac{\partial \dot\gamma}{\partial t} & 0 \\ \frac{\partial \dot\gamma}{\partial t} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \frac{b_{11}}{b_1} \dot\gamma^2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right]$$

The Cauchy Momentum Equation (in component form) is

x-component $\quad \rho \dfrac{\partial v_x}{\partial t} = -\dfrac{\partial \tau_{yx}}{\partial y} \qquad \left\{ \begin{array}{l} \text{since } v_y = v_z = 0 \\ \text{and } \tau_{xx} \neq f(x) \end{array} \right.$

Substituting for τ_{yx} from constitutive Eqn. gives

$$\rho \frac{\partial v_x}{\partial t} = + \frac{\partial}{\partial y} \left\{ b_1 \frac{\partial v_x}{\partial y} + b_2 \frac{\partial}{\partial t} \left(\frac{\partial v_x}{\partial y} \right) \right\}$$

Interchanging order gives

$$\boxed{\rho \frac{\partial v_x}{\partial t} = b_1 \left[\frac{\partial^2 v_x}{\partial y^2} + \frac{b_2}{b_1} \frac{\partial}{\partial t} \left(\frac{\partial^2 v_x}{\partial y^2} \right) \right]} \qquad (1)$$

b) Separation of Variables $\Rightarrow$ postulate a solution $v_x(y,t) = Y(y)T(t)$

Solution satisfies homogeneous B.C's $v_x(0) = 0$
$v_x(H) = 0$

Inhomogeneous I.C. $v_x(y,0) = U(y)$

Substitution into ① gives

$$\frac{\rho T'}{b_1\left[T + \frac{b_2}{b_1} T'\right]} = \frac{Y''}{Y} = -K_n^2 \quad \text{say}$$

<u>SPATIAL PART</u>

$$Y = \sum_{n=0}^{\infty} A_n \cos K_n y + B_n \sin K_n y$$

Using B.C's $y = 0,\ v_x = 0 \Rightarrow Y = 0 \Rightarrow A_n = 0$

$y = H,\ v_x = 0 \Rightarrow Y = 0 \Rightarrow \underline{\underline{K_n = \frac{n\pi}{H}}} \quad n = 1, 2..$

<u>TEMPORAL PART</u>

rearranging gives $T = -\left(\frac{b_2}{b_1} + \frac{\rho}{b_1 K_n^2}\right) T'$

define $\alpha_n = -\left(\frac{b_2}{b_1} + \frac{\rho}{b_1 K_n^2}\right)^{-1} \Rightarrow \frac{dT}{dt} - \alpha_n T = 0$

$$\Rightarrow \underline{\underline{T(t) = C_n e^{\alpha_n t}}}$$

Combining Solutions

$$\underline{\underline{v_x(y,t) = \sum_{n=1}^{\infty} \tilde{A}_n \sin\left(\frac{n\pi y}{H}\right) \exp(\alpha_n t)}}$$

where $\hat{A}_n \equiv (A_n C_n)$ and α_n is given above

$\hat{A}_n$ is determined from orthogonality and inhomogeneous initial data at $t = 0$

$$\int_0^H U(y) \sin(m\pi y/H)\, dy = \int_0^H \left(\sum_{n=1}^{\infty} \tilde{A}_n \sin(n\pi y/H)\right) \sin(m\pi y/H)\, dy$$
$$= \frac{\tilde{A}_n}{2} \delta_{mn}$$