4B.1 Axial Annular Flow with Inner Cylinder Moving Axially [OH]

Postulate: $v_r = v_\theta = 0$; $v_z = v_z(r) \rightarrow \underline{\underline{\tau}} = \underline{\underline{\tau}}(r)$

a) z-motion: $\frac{1}{r}\frac{d}{dr}(r\tau_{rz}) = 0$

$$\frac{dv_z}{dr} < 0 \rightarrow \quad \eta = m\left(-\frac{dv_z}{dr}\right)^{n-1} \rightarrow \tau_{rz} = m\left(-\frac{dv_z}{dr}\right)^n$$

$$\frac{d}{dr}\left[mr\left(-\frac{dv_z}{dr}\right)^n\right] = 0$$

b) Integrate: $mr\left(-\frac{dv_z}{dr}\right)^n = C_1 \quad // \times\frac{1}{mr}, \sqrt[n]{}, \int \ldots dr$

$$v_z = -\left(\frac{C_1}{m}\right)^{1/n} \frac{r^{1-1/n}}{1-1/n} + C_2 \quad , n \neq 1$$

c) Let $\sigma := 1/n$, $\xi := r/R$:

$$v_z = -\left(\frac{C_1}{m}\right)^\sigma \left[\frac{R^{1-\sigma}}{1-\sigma}\right] \xi^{1-\sigma} + C_2$$

B.C.s: 1) $v_z(\xi = \kappa) = V \rightarrow \left(\frac{C_1}{m}\right)^\sigma = \frac{V(1-\sigma)}{R^{1-\sigma}[1-\kappa^{1-\sigma}]}$

2) $v_z(\xi = 1) = 0 \rightarrow 0 = -\left(\frac{C_1}{m}\right)^\sigma \left[\frac{R^{1-\sigma}}{1-\sigma}\right] + C_2$

d)

$$\therefore \quad v_z = V\left[\frac{\xi^{1-\sigma} - 1}{\kappa^{1-\sigma} - 1}\right]$$

e) Ⓝ = lim $n \to 1, \sigma \to 1, m \to \mu$

$$V_z = \lim_{\epsilon \to 0} V\left[\frac{\xi^{\epsilon} - 1}{K^{\epsilon} - 1}\right] = V \lim_{\epsilon \to 0} \frac{\xi^{\epsilon} \ln \xi}{K^{\epsilon} \ln K} = V \frac{\ln \xi}{\ln K}$$

f)
$$F_z = -2\pi KRL\,\tau_{rz}\Big|_{r=KR} = -2\pi mKRL\left(-\frac{dV_z}{dr}\right)^{n}\Big|_{r=KR}$$

$$= -2\pi mKRL\left\{\frac{-V}{K^{1-\sigma}-1}\right\}\frac{(1-\sigma)(KR)^{-\sigma}}{R^{1-\sigma}}$$

$$= -2\pi mKRL\left\{\frac{-V(1-\sigma)}{(K-K^{\sigma})R}\right\}^{n}$$

g) According to the def^{n} in §3.7 (p.155), since $\underline{\underline{\dot{\gamma}}}$ has only one component, this is a shear flow.

h)
$$Q = \int_{KR}^{R} V_z 2\pi r\,dr = \left(\frac{2\pi R^2 V}{K^{1-\sigma}-1}\right)\int_{K}^{1}(\xi^{2-\sigma} - \xi)\,d\xi$$

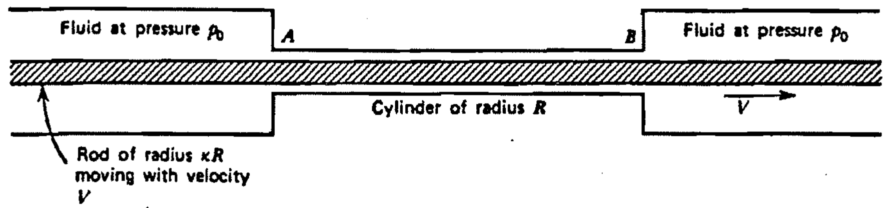
$$= \left(\frac{2\pi R^2 V}{K^{1-\sigma}-1}\right)\left[\frac{1-K^{3-\sigma}}{3-\sigma} - \frac{1-K^2}{2}\right]$$

$\epsilon := 1-\sigma$

i)
$$Q\,(\text{Newtonian}) = \lim_{\epsilon \to 0}\left[\frac{2\pi R^2 V}{K^{\epsilon}-1}\right]\left[\frac{-2K^{2+\epsilon} + 2K^2 - \epsilon + \epsilon K^2}{2(2+\epsilon)}\right]$$

$$= \pi R^2 V\left[\frac{-2K^2 \ln K + K^2 - 1}{2\ln K}\right] = \frac{\pi R^2 V}{2\ln K}\left[K^2(1-2\ln K) - 1\right]$$

4B.1 Axial Annular Flow with Inner Cylinder Moving Axially —(Power-Law).



Postulate $\quad v_r = v_\theta = 0$

$v_z = v_z(r)$ only

$\therefore \underline{\underline{\tau}} = \underline{\underline{\tau}}(r)$

a. The z-component of the eqn. of motion can be written as,

$$\frac{1}{r}\frac{d}{dr}(r\tau_{rz}) = 0 \qquad \text{—(1)}$$

$$\frac{dv_z}{dr} < 0 . \quad \therefore \left|\frac{dv_z}{dr}\right| = -\frac{dv_z}{dr} .$$

For a power-law fluid:

$$\eta = m\left(-\frac{dv_z}{dr}\right)^{n-1}$$

$$\tau_{rz} = -\eta\frac{dv_z}{dr} = m\left(-\frac{dv_z}{dr}\right)^{n} \qquad \text{—(2)}$$

Substituting (2) into (1), we get

$$\frac{d}{dr}\left[mr\left(-\frac{dv_z}{dr}\right)^{n}\right] = 0 \qquad \text{—(3)}$$

4B.1 (Contd.)

b. Integrating eqn. (3) twice, we get

$$v_z = -\left(\frac{c_1}{m}\right)^{1/n}\left[\frac{r^{1-1/n}}{(1-1/n)}\right] + c_2\ , \quad \text{for } n \neq 1. \qquad \text{—(4a)}$$

or $$v_z = -\left(\frac{c_1}{m}\right)^{s}\left[\frac{r^{1-s}}{1-s}\right] + c_2\ , \quad \text{where } s = \frac{1}{n}. \qquad \text{—(4b)}$$

c. Let $\xi = r/R$

$$v_z = -\left(\frac{c_1}{m}\right)^{s}\left(\frac{R^{1-s}}{1-s}\right)\xi^{(1-s)} + c_2 \qquad \text{—(5)}$$

The boundary-conditions we need to use to determine c_1, c_2 are:

$$v_z(\xi = K) = V \quad \text{(no-slip)} \qquad \text{—(6a)}$$

$$v_z(\xi = 1) = 0 \quad (\ \text{"}\ ) \qquad \text{—(6b)}$$

d. Using conditions (6) we determine that,

$$c_2 = \left(\frac{c_1}{m}\right)^{s}\frac{R^{(1-s)}}{(1-s)}, \quad \text{and}$$

$$\left(\frac{c_1}{m}\right)^{s} = \frac{V\ (1-s)}{R^{(1-s)}\left[1-K^{1-s}\right]}$$

$$\therefore \quad \frac{v_z}{V} = \left[\frac{\xi^{1-s}-1}{K^{1-s}-1}\right] \qquad \text{—(7)}$$

4B.1 (Contd.)

For a Newtonian fluid, $n = s = 1$.

We can find the expression for the velocity by finding the limit of the expression on the right-side of eqn. (7) as $s \to 1$.

Let $(1 - s) = \epsilon$. $\therefore \epsilon \to 0$ as $s \to 1$.

$$\frac{v_z}{V} = \frac{\xi^{\epsilon} - 1}{K^{\epsilon} - 1}$$

$$\frac{v_z}{V} = \lim_{\epsilon \to 0} \frac{\xi^{\epsilon} - 1}{K^{\epsilon} - 1} = \lim_{\epsilon \to 0} \frac{\xi^{\epsilon} \ln \xi}{K^{\epsilon} \ln K} = \frac{\ln \xi}{\ln K} \qquad \text{—(8)}$$

{This result may also be obtained by integrating eqn. (3) for this particular case, but that isn't what you're asked to do in (e).}

f.

Force acting on the wire between A and B $= -(2\pi KRL)\, \tau_{rz}\big|_{r=KR}$ —(9)

From eqn. (2) $\tau_{rz} = m\left(-\frac{\partial v_z}{\partial r}\right)^n$

$$v_z = \left\{\frac{V}{K^{1-s} - 1}\right\}\left\{\left(\frac{r}{R}\right)^{1-s} - 1\right\} \; ; \quad \frac{\partial v_z}{\partial r}\Big|_{r=KR} = \left\{\frac{V}{K^{1-s} - 1}\right\} \frac{(1-s)(KR)^{-s}}{R^{(1-s)}}$$

$\therefore$ Force acting on the wire $= -(2\pi m KRL)\left\{\frac{-V(1-s)}{(K - K^{s})R}\right\}^n$ —(10)

Force acting on the wire (Newtonian fluid) $= (2\pi m KRL)\left\{\frac{V}{KR \ln K}\right\}$ —(11)

g. For this flow $\dot{\gamma}$ has only one component $\dot{\gamma}_{rz}$ (or $\dot{\gamma}_{zr}$). Therefore according to the definition in § 3.7 (p.155), this is a shear flow.

4B.1 (Contd.)

h. Volume flow rate through the annular region, $Q = \int_{KR}^{R} (2\pi r) v_z \, dr$ —(12)

$$\therefore Q = \left(\frac{2\pi R^2 V}{K^{1-s} - 1}\right) \int_K^1 (\xi^{2-s} - \xi)\, d\xi = \left(\frac{2\pi R^2 V}{K^{1-s} - 1}\right)\left[\frac{1-K^{3-s}}{(3-s)} - \frac{1-K^2}{2}\right] \quad (13)$$

i. We need to show how the expression obtained in h (eqn. 13) simplifies for $s = 1$. Let $s = 1 - \epsilon$ where $\epsilon \ll 1$ (i.e. $\epsilon \to 0$)

$$\therefore Q\,(\text{Newtonian}) = \lim_{\epsilon \to 0} \left[\frac{2\pi R^2 V}{K^{\epsilon} - 1}\right]\left[\frac{-2K^{2+\epsilon} + 2K^2 - \epsilon + \epsilon K^2}{2(2+\epsilon)}\right]$$

$$Q\,(\text{Newtonian}) = (\pi R^2 V)\left[\frac{-2K^2 \ln K + K^2 - 1}{2 \ln K}\right] = \frac{\pi R^2 V}{2 \ln K}\left[K^2(1 - 2\ln K) - 1\right] \quad (14)$$

{This result can also be obtained by substituting the expression obtained in e (eqn. 8), for v_z, into eqn. (12) and integrating as in (h).}

4B.4 Distributor Design (Power Law) [JDS]

a) ΔQ = flow out of slit per unit length

$= Q_0/L$, since efflux is uniform

mass bal: $Q(z) = Q_0 - \int_0^z \Delta Q \, dz = Q_0(1 - z/L)$

$$Q_0(1 - z/L) = \frac{\pi R^3 n}{3n+1}\left(-\frac{R}{2m}\frac{dp}{dz}\right)^{1/n} \qquad (4B.4\text{-}2)$$

$$-\int_p^{p_a} dp' = \frac{2m}{R}\left[\frac{(3n+1)Q_0}{\pi R^3 n}\right]^n \int_z^L (1 - z/L)^n dz \quad ; \zeta := z/L$$

$$p - p_a = \frac{2mL}{R(n+1)}\left[\frac{(3n+1)Q_0}{\pi R^3 n}\right]^n (1-\zeta)^{n+1} \qquad (4B.4\text{-}3)$$

b) $\frac{Q}{W} = \frac{B^2 n}{2(2n+1)}\left[\frac{(p-p_a)B}{2 l m}\right]^{1/n}$, from 4.2-1; note B is defined differently her

use $V = \frac{Q}{WB} \rightarrow \ell(\zeta) = \frac{B(p-p_a)}{2m}\left[\frac{nB/2}{(2n+1)V}\right]^n$

c)

use (4B.4-3):

$$\ell(\zeta) = \frac{BL}{R(n+1)}\left[\frac{(3n+1)Q_0}{\pi R^3 n}\right]^n \left[\frac{nB/2}{V(2n+1)}\right]^n (1-\zeta)^{n+1}$$

which leads to (4B.4-5).

$\boxed{4B.4}$ a) $\Delta Q \equiv$ flow out of slit per unit length of slit
$= Q_0/L$, since efflux is uniform

mass balance:

$$Q(z) = Q_0 - \int_0^z \Delta Q\, dz = Q_0(1 - z/L)$$

$$Q_0(1 - z/L) = \frac{\pi R^3 n}{3n+1}\left(-\frac{R}{2m}\frac{dp}{dz}\right)^{1/n}$$

$$-\frac{dp}{dz} = \left[\frac{(3n+1)Q_0(1-z/L)}{\pi R^3 n}\right]^n \frac{2m}{R}$$

$$-\int_p^{p_a} dp = \frac{2m}{R}\left[\frac{(3n+1)Q_0}{\pi R^3 n}\right]^n \int_z^L (1 - z/L)^n dz \qquad ; \ \zeta := z/L$$

$$\underline{\underline{p - p_a = \frac{2mL}{R(n+1)}\left[\frac{(3n+1)Q_0}{\pi R^3 n}\right]^n (1-\zeta)^{n+1}}}$$

b) $\frac{Q}{W} = \frac{B^2 n}{2(2n+1)}\left[\frac{(p-p_a)B}{2\ell m}\right]$, from 4.2-1 , note B is defined differently here

use $V = \frac{Q}{WB} \rightarrow \ell(\zeta) = \frac{B(p-p_a)}{2m}\left[\frac{nB/2}{(2n+1)V}\right]^n$

$$\ell(\zeta) = \frac{BL}{R(n+1)}\left[\frac{(3n+1)Q_0}{\pi R^3 n}\right]^n \left[\frac{nB/2}{V(2n+1)}\right]^n (1-\zeta)^{n+1}$$

$$\boxed{\ell(\zeta) = \frac{BL}{R(n+1)}\left[\frac{(3n+1)}{(2n+1)}\cdot\frac{B}{2\pi R^3}\right]^n \left(\frac{Q_0}{V}\right)^n (1-\zeta)^{n+1}}$$

4B.7 Flow in Circular Tubes and Slits (Any Generalized Newtonian Fluid) [RBB]

a. $$Q = \int_0^{2\pi}\int_0^R v_z\, r\, dr\, d\theta = 2\pi\int_0^R v_r\, r\, dr$$

$$= 2\pi\left[\frac{v_r r^2}{2}\Big|_0^R - \int_0^R \frac{r^2}{2}\frac{dv_z}{dr}\,dr\right]$$

$$= -\pi\int_0^R r^2\frac{dv_z}{dr}\,dr \quad \text{since } v_r = 0 \text{ at } r = R$$

$$= -\pi\left[\frac{r^3}{3}\frac{dv_z}{dr}\Big|_0^R - \int_0^R \frac{r^3}{3}\frac{d}{dr}\left(\frac{dv_z}{dr}\right)dr\right]$$

$$= \frac{\pi R^3}{3}\dot{\gamma}_R - \frac{\pi}{3}\int_0^{\dot{\gamma}_R}[r(\dot{\gamma})]^3\, d\dot{\gamma}$$

Now from $\tau_{rz} = -\eta\,(dv_z/dr)$ we get for tube flow

$$\tau_R \cdot \frac{r}{R} = +\eta\dot{\gamma}$$

whence $r = (R/\tau_R)\,\eta\dot{\gamma}$ (with $\eta = \eta(\dot{\gamma})$). We now eliminate r in favor of $\eta(\dot{\gamma})$:

$$Q = \frac{\pi R^3}{3}\dot{\gamma}_R - \frac{\pi}{3}\left(\frac{R}{\tau_R}\right)^3\int_0^{\dot{\gamma}_R}[\eta(\dot{\gamma})\,\dot{\gamma}]^3\, d\dot{\gamma}$$

and $\dot{\gamma}_R$ is given by $\dot{\gamma}_R = \tau_R/\eta(\dot{\gamma}_R)$ -- which has to be solved for $\dot{\gamma}_R$.

For a power-law model $\tau_R = m\dot{\gamma}_R^{\,n}$ and $\dot{\gamma}_R = (\tau_R/m)^{1/n}$.

Then

$$Q = \frac{\pi R^3}{3}\left(\frac{\tau_R}{m}\right)^{\frac{1}{n}} - \frac{\pi R^3}{3}\left(\frac{m}{\tau_R}\right)^3 \int_0^{\dot{\gamma}_R} \dot{\gamma}^{3n}\, d\dot{\gamma}$$

$$= \frac{\pi R^3}{3}\left(\frac{\tau_R}{m}\right)^{\frac{1}{n}} - \frac{\pi R^3}{3}\left(\frac{m}{\tau_R}\right)^3 \frac{\dot{\gamma}_R^{\,3n+1}}{3n+1}$$

$$= \frac{\pi R^3}{3}\left[\left(\frac{\tau_R}{m}\right)^{\frac{1}{n}} - \left(\frac{m}{\tau_R}\right)^3 \left(\frac{\tau_R}{m}\right)^{\frac{1}{n}+3} \frac{1}{3n+1}\right]$$

$$= \frac{\pi R^3}{3}\left(\frac{\tau_R}{m}\right)^{\frac{1}{n}} \left(1 - \frac{1}{3n+1}\right)$$

$$= \frac{\pi R^3}{(1/n)+3}\left(\frac{\tau_R}{m}\right)^{\frac{1}{n}}$$

which agrees with the next-to-last line of Eq. 4.2-9.

b. $$Q = \int_0^W \int_{-B}^{+B} v_z(x)\, dx\, dy = 2W \int_0^B v_z(x)\, dx$$

$$= 2W\left[x v_z(x)\Big|_0^B - \int_0^B x \frac{dv_z}{dx}\, dx\right]$$

$$= -2W \int_0^B x \frac{dv_z}{dx}\, dx$$

$$= -2W\left[\frac{x^2}{2}\frac{dv_z}{dx}\Big|_0^B - \int_0^B \frac{x^2}{2}\frac{d}{dx}\left(\frac{dv_z}{dx}\right) dx\right]$$

$$= WB^2\dot{\gamma}_B - W\int_0^{\dot{\gamma}_B} [x(\dot{\gamma})]^2\, d\dot{\gamma}$$

where $\dot{\gamma} = -dv_z/dx$, and $\dot{\gamma}_B = -(dv_z/dx)\big|_{x=B}$.

The for the slit we have

$$\tau_{xz} = -\eta \frac{dv_z}{dx}$$

and for $0 \leq x \leq B$ this is the same as:

$$\tau_B \frac{x}{B} = +\eta \dot{\gamma} \qquad \left(\tau_B = \tau_{xz}\big|_{x=B}\right)$$

so that $\quad x = \left(\frac{B}{\tau_B}\right)\eta\dot{\gamma}$

Then $\quad Q = WB^2\dot{\gamma}_B - W\left(\frac{B}{\tau_B}\right)^2 \int_0^{\dot{\gamma}_B} [\eta(\dot{\gamma})\dot{\gamma}]^2 d\dot{\gamma}$

where $\dot{\gamma}_B$ is given in terms of the pressure difference by $\tau_B = \eta(\dot{\gamma}_B)\dot{\gamma}_B$, with $\tau_B = (\mathcal{P}_0 - \mathcal{P}_L)B/L$.

For the power law $\tau_B = m\dot{\gamma}_B^n$ or $\dot{\gamma}_B = (\tau_B/m)^{1/n}$, and

$$Q = WB^2\left(\frac{\tau_B}{m}\right)^{1/n} - W\left(\frac{B}{\tau_B}\right)^2 \int_0^{\dot{\gamma}_B} (m\dot{\gamma}^n)^2 d\dot{\gamma}$$

$$= WB^2\left(\frac{\tau_B}{m}\right)^{1/n} - WB^2\left(\frac{m}{\tau_B}\right)^2 \frac{\dot{\gamma}_B^{2n+1}}{2n+1}$$

$$= WB^2\left[\left(\frac{\tau_B}{m}\right)^{1/n} - \left(\frac{m}{\tau_B}\right)^2\left(\frac{\tau_B}{m}\right)^{\frac{1}{n}+2}\frac{1}{2n+1}\right]$$

$$= WB^2\left(\frac{\tau_B}{m}\right)^{1/n}\left[1 - \frac{1}{2n+1}\right]$$

$$= 2WB^2\left(\frac{\tau_B}{m}\right)^{1/n}\frac{1}{(1/n)+2} \qquad \leftarrow \text{Eq. A of Table 4.5-2.}$$