

MASSACHUSETTS INSTITUTE OF TECHNOLOGY  
DEPARTMENT OF MECHANICAL ENGINEERING  
CAMBRIDGE, MASSACHUSETTS 02139  
**2.29 NUMERICAL FLUID MECHANICS — SPRING 2015**

**QUIZ 1**

Wednesday, March 11, 2015

The goals of quiz 1 are to: (i) ask some general higher-level questions to ensure you understand broad concepts and are able to discuss such concepts and issues with others familiar with numerical fluid mechanics; (ii) show that you understand methods and schemes that you learned and that you can apply them in idealized computational problems; and (iii), evaluate if you can read numerical codes and recognize what they accomplish. Partial credit will be given to partial answers (e.g. accurate descriptions in words of computations required but not carried out).

**A. Shorter Concept Questions**

**Problem I (24 points)**

*Briefly answer only 6 of the following 7 questions* (a few words to a few sentences is enough, with or without a few equations depending on the question). If you answer 7 questions, we will give a bonus.

- a) In a floating point real number representation, *briefly* explain why round-off errors increase with the magnitude of the number?
- b) Provide two examples of computations that are sensitive to round-off errors and very briefly state why they are sensitive.
- c) Using the bisection method to determine the root of a nonlinear function is not very efficient. Nonetheless, provide one example when the use of the bisection method can accelerate the convergence of another method.
- d) Using the Taylor expansion,  $f(x_{i+1}) = f(x_i) + \Delta x f'(x_i) + \frac{\Delta x^2}{2!} f''(x_i) + R_2$ , obtain an open method to solve  $f(x) = 0$  that can have a higher order of convergence than Newton-Raphson but that does not evaluate second-order derivatives (*Hint*: remember why and how the Secant method was derived).
- e) To increase the accuracy of a solver for a linear partial differential equation (PDE), you decide to increase the order of the finite-difference scheme. Briefly explain one effect of this change on the discrete linear system to be solved.
- f) Consider an iterative scheme,  $\mathbf{x}^{k+1} = \mathbf{B} \mathbf{x}^k + \mathbf{c}$ ,  $k = 0, 1, 2, \dots$ , used to solve a linear system. Briefly explain why the norm of  $\mathbf{B}$  needs to be smaller than one.
- g) Provide an example and one property of parabolic PDEs.

## B. Computational Schemes and Idealized Computational Problems

### Problem II: Estimation of Errors and Conditioning in Navier-Stokes Equations (18 points)

Consider the incompressible Navier-Stokes momentum equations,

$$\frac{\partial \rho \vec{v}}{\partial t} + \nabla \cdot (\rho \vec{v} \vec{v}) = -\nabla p + \mu \nabla^2 \vec{v} + \rho \vec{g}.$$

We will employ Cartesian coordinates and assume two-dimensional flows.

- Assuming a relative error  $\vec{\varepsilon} = (\varepsilon_u, \varepsilon_v)$  for the velocity  $\vec{v} = (u, v)$ , estimate the magnitude of the corresponding relative error in the advection (momentum) flux  $f_a(u, v) = \rho uv$ .
- Based on the results in a), *briefly* compare the conditioning of this advection flux  $f_a(u, v)$  to that of a scalar diffusion flux, e.g.  $f_d(u) = \mu \frac{\partial u}{\partial y}$ .

Consider now the full system solve for a discretized form of the above Navier-Stokes equations.

- First, consider a much simpler system:

$$\begin{bmatrix} 50 & 50 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

What is a condition number of this system? Is the system well-conditioned?

Provide one simple way to improve the conditioning of the system.

- If at each time-step, the discretized form of the Navier-Stokes equations leads to a coupled linear system solve for the two velocities and the pressure, utilize your results in c) to *very briefly* state a simple way to reduce the possible ill-conditioning of the coupled system.

### Problem III: Consistency and Accuracy of a Finite-Difference Scheme (19 points)

Consider the one-dimensional heat conduction PDE with constant coefficient  $\alpha$ ,

$$\frac{\partial T}{\partial t} - \alpha \frac{\partial^2 T}{\partial x^2} = 0. \quad (1)$$

- Discretizing time as  $t = n \Delta t$ ,  $n = 0, 1, \dots$  and space as  $x = i \Delta x$ ,  $i = 1, \dots, m$ , show that the following finite-difference scheme, with  $s = \alpha \Delta t / \Delta x^2$ ,

$$\hat{T}_i^{n+1} = s \hat{T}_{i-1}^n + (1 - 2s) \hat{T}_i^n + s \hat{T}_{i+1}^n \quad (2)$$

is consistent with the above heat conduction PDE (1). (*Hint*: substitute the unknown exact solution  $T(x, t)$  of (1) into each of the four terms of (2), expand each of them in Taylor series about the  $(i, n)$ <sup>th</sup> computational node, and obtain the dominant truncation error of the finite-difference scheme (2).)

- What is the numerical value of  $s$  that leads to the highest degree of accuracy (in time or in space)?

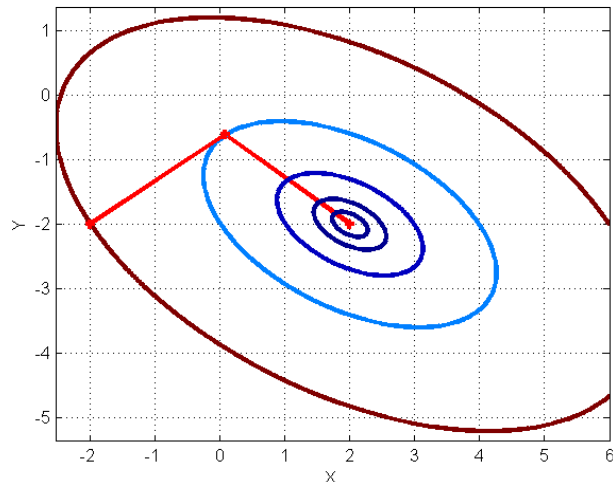
**Problem IV: Conjugate Gradient Method (19 points)**

The conjugate gradient algorithm to solve a linear system  $\mathbf{Ax} = \mathbf{b}$  for a symmetric positive-definite matrix  $\mathbf{A}$  is given by:

$$\begin{aligned} \mathbf{v}_0 &= \mathbf{r}_0 = \mathbf{b} - \mathbf{Ax}_0 && \text{Initialization} \\ \text{Do} &&& \\ \alpha_i &= \frac{\mathbf{v}_i^T \mathbf{r}_i}{\mathbf{v}_i^T \mathbf{Av}_i} && \text{Optimal Step Length} \\ \mathbf{x}_{i+1} &= \mathbf{x}_i + \alpha_i \mathbf{v}_i && \text{Approximate Solution} \\ \mathbf{r}_{i+1} &= \mathbf{r}_i - \alpha_i \mathbf{Av}_i && \text{New Residual} \\ \beta_i &= -\frac{\mathbf{v}_i^T \mathbf{Ar}_{i+1}}{\mathbf{v}_i^T \mathbf{Av}_i} && \text{Step Length \&} \\ \mathbf{v}_{i+1} &= \mathbf{r}_{i+1} + \beta_i \mathbf{v}_i && \text{New Search Direction} \\ \text{Until stop criterion holds} &&& \end{aligned}$$

where  $\mathbf{r}_i = \mathbf{b} - \mathbf{Ax}_i$  is the residual at step  $i$ .

- Using the ‘‘Approximate Solution’’ stage,  $\mathbf{x}_{i+1} = \mathbf{x}_i + \alpha_i \mathbf{v}_i$ , briefly derive the expression for the optimal step length  $\alpha_i$  by enforcing  $\mathbf{Ax}_{i+1} = \mathbf{b}$  but only along the search direction  $\mathbf{v}_i$ . In other words, this optimal  $\alpha_i$  sets the next residual  $\mathbf{r}_{i+1}$  to zero along  $\mathbf{v}_i$ , i.e.  $\mathbf{v}_i^T \mathbf{r}_{i+1} = 0$ .
- Using the ‘‘New Search Direction’’ stage,  $\mathbf{v}_{i+1} = \mathbf{r}_{i+1} + \beta_i \mathbf{v}_i$ , briefly derive the expression for the optimal step length  $\beta_i$  by enforcing  $\mathbf{A}$ -conjugate search directions  $\mathbf{v}_i$ , i.e.  $\mathbf{v}_i^T \mathbf{Av}_{i+1} = 0$ .
- With the result obtained in a), i.e.  $\mathbf{v}_i^T \mathbf{r}_{i+1} = 0$  (and thus  $\mathbf{v}_{i-1}^T \mathbf{r}_i = 0$  by recursion), and the results obtained in b), i.e.  $\mathbf{v}_i^T \mathbf{Av}_{i+1} = 0$  (and thus  $\mathbf{v}_{i-1}^T \mathbf{Av}_i = 0$  by recursion), utilize the ‘‘New Search Direction’’ stage,  $\mathbf{v}_i = \mathbf{r}_i + \beta_{i-1} \mathbf{v}_{i-1}$ , and ‘‘New Residual’’ stage,  $\mathbf{r}_{i+1} = \mathbf{r}_i - \alpha_i \mathbf{Av}_i$ , to show that the successive residuals are orthogonal, i.e.  $\mathbf{r}_i^T \mathbf{r}_{i+1} = 0$ .
- Building on c), one can show that each residual is orthogonal to all previous residuals, i.e.  $\mathbf{r}_i^T \mathbf{r}_k = 0, \forall k < i$  (no need to show this). If the matrix  $\mathbf{A}$  is of size  $n$ , what can you then say about  $\mathbf{r}_{n+1}$ ?
- Using the above, *very briefly explain* why the last conjugate search direction  $\mathbf{v}_{n-1}$  points always to the minimum (regardless of the first direction  $\mathbf{v}_0$ ). The figure on the right (which shows a conjugate gradient solution for a 2-by-2 linear system) could be useful.



## C. Numerical Code Evaluations

### Problem V: *Finite Difference Solution using MCVNCD*<sup>®</sup> (20 points)

Read the following MATLAB function and answer the questions that follow.

```
function [phi] = PDE_Solve2(Lx, Ly, Nx, Ny, a, v, f, bcs_w, bcs_s, bcs_e)
% INPUTS:
% Lx:          Size of domain along x,          size(Lx) = 1
% Ly:          Size of domain along y,          size(Ly) = 1
% Nx-1:        Number of interior nodes along x, size(Nx) = 1
% Ny:          Number of interior nodes along y, size(Ny) = 1
% a:           Equation parameter,              size(a) = 1
% v:           Equation parameter,              size(v) = 1
% f:           Function handle;                 Inputs: (x,y)
% bcs_w:        West boundary conditions,        size(bcs_w) = [Ny+1, 1]
% bcs_s:        South boundary conditions,        size(bcs_s) = [1, Nx-1]
% bcs_e:        East boundary conditions,         size(bcs_e) = [Ny+1, 1]

Max_Iter = 2000; EPS = 1e-10;

dx = Lx/Nx;
dy = Ly/Ny;
A = a/dx^2;
V = v/dy;

[X, Y] = meshgrid(0:dx:Lx, 0:dy:Ly);
F = f(X, Y);

% Set initial guess and boundary conditions.
phi = zeros(Ny+1, Nx+1);
phi(:,1) = bcs_w;
phi(:,end) = bcs_e;
phi(1, 2:end-1) = bcs_s;

% Initialize iteration
Iter_Count = 0;
Iter_Diff = 1e10;

% Start iteration
while (Iter_Count < Max_Iter) && (Iter_Diff > EPS)
    Iter_Diff = 0;
    for i = 2 : Nx
        for j = 2 : Ny+1
            temp = (A*phi(j,i-1) + A*phi(j,i+1) + V*phi(j-1,i) + F(j,i))/(2*A+V);
            if (abs(phi(j,i)-temp)>Iter_Diff)
                Iter_Diff = abs(phi(j,i)-temp);
            end;
            phi(j,i) = temp;
        end;
    end;
    Iter_Count = Iter_Count + 1;
end;
```

- The code solves a PDE by a finite difference method. Please write down the PDE and its finite difference approximation at an interior node.
- What is the name of the linear solve algorithm being used to solve the linear system coming out of the discretization?

c) If we align the unknowns (values of  $\phi$  at interior nodes) in a column vector as

$$[\phi_{2,2}, \phi_{3,2}, \dots, \phi_{N_y+1,2}, \phi_{2,3}, \dots, \phi_{N_y+1,3}, \dots, \phi_{2,N_x}, \dots, \phi_{N_y+1,N_x}]^T$$

(note that  $\phi_{j,i}$  here corresponds to “phi(j,i)” in the code), the  $\mathbf{A}$  matrix of this linear system is also determined. Is  $\mathbf{A}$  strictly diagonally dominant? If yes, please give a brief proof. If not, please specify which rows of  $\mathbf{A}$  violate the strict diagonal dominance (Or you can specify which nodes correspond to rows of  $\mathbf{A}$  that violate the strict diagonal dominance).

d) How many nonzero diagonals does  $\mathbf{A}$  have? What is the full bandwidth of  $\mathbf{A}$ ?

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## 2.29 Numerical Fluid Mechanics

Spring 2015

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