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Similarity transformation methods in the analysis of the two dimensional steady compressible laminar boundary layer

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ABSTRACT

The system of equations in a steady, compressible, laminar boundary layer is composed of four fundamental equations. Those are: the continuity equation, the momentum equation, the energy equation, and the equation of state. The solutions of these equations, when solved simultaneously for a 2-dimensional boundary layer, are: the velocity in the x and y direction (u and v), the pressure (p) and the density (ρ). The system of equations is a system of partial differential equations (PDE) and is usually difficult to solve. Therefore, sophisticated transformation methods, called similarity transformations are introduced to convert the original partial differential equation set to a simplified ordinary differential equation (ODE) set. The solutions of this ordinary differential equation set are usually nondimensionalized velocities and temperature. By principle, these ordinary equations are coupled mathematically and usually can be solved by numerical methods. However, with further appropriate assumptions related to the transport properties (e.g. Prandtl number), and flow conditions (e.g. Mach number, geometry around flow), these ODEs can be uncoupled mathematically or can have simpler forms, almost similar to the forms obtained from the incompressible boundary layer analysis. (e.g. Blasius solution, Falkner-Skan equation). Hence, the simplified ODE set makes it possible to get the solution from the already existing solutions of the incompressible analysis and also reduces the computing time in the numerical analysis.

In this paper, three different transformation methods will be described. A detailed derivation of the generalized (Levy-Ilingworth) transformation method and the appropriate assumptions made during the derivation will be explained. The Howarth transformation and the Illingworth-Stewartson transformation will be described briefly.

INTRODUCTION

The system of equations in the incompressible boundary layer with forced convection, is a PDE system composed of the continuity, the momentum, and the energy equations. These simultaneous equations can be reduced to two ODEs using similarity transformation. In this case, continuity equation and momentum equation are reduced to a single ODE and energy equation is reduced to another ODE.

Compared with the incompressible boundary layer analysis, the effect of compressibility on the entire velocity and temperature field should be considered. As a result, the system of equations in compressible boundary layer is a more complicated PDE system, composed of the continuity equation, the momentum equation, the energy equation and an equation of state.

SYSTEM OF EQUATIONS OF COMRESSIBLE BOUNDARYLAYER

The system of governing equations to be solved for a two-dimensional, steady, compressible, laminar boundary layer without body forces and bulk heat transfer is as follows:

GOVERNING EQUATIONS

Continuity equation

$$\frac{\partial}{\partial x}(\rho u)+\frac{\partial}{\partial y}(\rho v)=0 \quad (1)$$

Momentum equation

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=-\frac{1}{\rho}\frac{\partial P}{\partial x}+\frac{1}{\rho}\frac{\partial}{\partial y}\left(\mu\frac{\partial u}{\partial y}\right)$$
$$\frac{\partial P}{\partial y}=0 \qquad (2)$$

Energy equation

$$u\frac{\partial h}{\partial x}+v\frac{\partial h}{\partial y}=\frac{u}{\rho}\frac{\partial P}{\partial x}+\frac{1}{\Pr}\frac{1}{\rho}\frac{\partial}{\partial y}\left(\mu\frac{\partial h}{\partial y}\right)+v\left(\frac{\partial u}{\partial y}\right)^2 \qquad (3)$$

An equation of state

$$\mathrm{p}=\rho \mathrm{RT} \qquad (4)$$

where,

x : Direction along the surface creating the boundary layer
y : Direction normal to the surface
u : Velocity in the x direction
v : Velocity in the y direction
ρ : Density
p : Pressure
μ : Viscosity
ν : Kinematic viscosity
p : Pressure
h : Enthalpy
R : Gas constant

Comparing the energy equation (3) to the energy equation (A.47) used in incompressible boundary layer with forced convection shown in Appendix.4, the first term in the energy equation in (3) is retained, which is the compressive work term $\frac{u}{\rho}\frac{\partial P}{\partial x}$. The second term on the right hand side of the energy equation represents the diffusion of heat transferred to the fluid or generated within the fluid. The third term represents the heat generated due to viscous stresses within the fluid, i.e., viscous dissipation.

BOUNDARY CONDITIONS

These boundary conditions at the surface, i.e., $y=0$ are given by the no-slip velocity condition with or without mass transfer or heat transfer.

$$u(0)=0 \quad v(0)=v(x) \quad h(0)=h(x) \quad \frac{\partial h}{\partial y}=0 \qquad (5)$$

At the edge of the boundary layer, the viscous flow inside the boundary layer is required to smoothly transition into the inviscid flow outside the boundary layer.

$$u(y\to\infty)\to U_e(x), \quad h(y\to\infty)\to h_e(x) \qquad (6)$$

where, the subscript e represents condition at the edge of the boundary layer.

NONDIMENSIONAL FORM OF THE EQUATIONS

Introducing the non-dimensional variables:

$$\overline{u}=\frac{u}{U_e} \quad \overline{\mathrm{v}}=\frac{\mathrm{v}}{\mathrm{U_e}} \quad \overline{\mathrm{y}}=\frac{\mathrm{y}}{\mathrm{L}} \quad \overline{\mathrm{x}}=\frac{\mathrm{x}}{\mathrm{L}} \qquad (7)$$

$$\overline{h}=\frac{h}{h_e} \quad \overline{\mu}=\frac{\mu}{\mu_\mathrm{e}} \quad \overline{\mathrm{P}}=\frac{\mathrm{P}}{\rho_\mathrm{e}\mathrm{U}_\mathrm{e}^2} \quad \overline{\rho}=\frac{\rho}{\rho_e} \qquad (8)$$

then, the original equations (1)~(4) become:

$$\frac{\partial(\overline{\rho u})}{\partial\overline{x}}+\frac{\partial(\overline{\rho v})}{\partial\overline{y}}=0 \qquad (9)$$

$$\overline{u}\frac{\partial\overline{u}}{\partial\overline{x}}+\overline{v}\frac{\partial\overline{u}}{\partial\overline{y}}=-\frac{1}{\rho}\frac{\partial\overline{P}}{\partial\overline{x}}+\frac{1}{\mathrm{Re}}\frac{\partial}{\partial\overline{y}}\left(\overline{\mu}\frac{\partial\overline{u}}{\partial\overline{y}}\right) \qquad (10)$$

$$\overline{u}\frac{\partial\overline{h}}{\partial\overline{x}}+\overline{v}\frac{\partial\overline{h}}{\partial\overline{y}}=(\gamma-1)M_e^2\frac{\overline{u}}{\overline{\rho}}\frac{\partial\overline{P}}{\partial\overline{x}}+\frac{1}{\Pr}\frac{1}{\mathrm{Re}}\frac{\partial}{\partial\overline{y}}\left(\overline{\mu}\frac{\partial\overline{h}}{\partial\overline{y}}\right)$$
$$+\frac{(\gamma-1)M_e^2}{\mathrm{Re}}\overline{v}\left(\frac{\partial\overline{u}}{\partial\overline{y}}\right)^2$$

(11)

where,

$\mathrm{Re}=\frac{\rho_e U_e L}{\mu_e}$: Reynolds number

$$(\gamma-1)M_e^2=\frac{U_e^2}{h_e}$$

$\gamma=\frac{C_p}{C_v}$: Specific heat ratio

$M_e=\frac{U_e}{c}$: Mach number

C_p : Specific heat at constant pressure

C_v : Specific heat at constant volume

In the non-dimensional energy equation (11), the first term, i.e., the work due to compression and the third term, i.e., the heat generated by viscous dissipation become increasingly important as the Mach number of the external flow increases.

BASIC ASSUMPTONS IN THE COMPRESSIBLE BOUNDAY LAYER

In the PDE system composed of equations (1)~(4), the influence of compressibility is first contained directly in the density terms ρ in the continuity equation (1), and more indirectly as a variable coefficient in the momentum equation (2) and energy equation (3). The second influence of compressibility is to produce temperature variations that are too large to permit the assumption of constant properties μ and k.

It is common to use the energy equation written in terms of enthalpy h in compressible problems instead of k as shown in the energy equation (3), in which the Prandtl number (Pr) is shown instead of $k = \frac{\mu c_p}{\Pr}$. Therefore, the added complexity with compressible, laminar boundary layer problems is centered on variable ρ, μ, and Pr.

From an equation of state, the density is a function of temperature and pressure, i.e., $\rho = \rho(T, P)$. However, the pressure is assumed constant across the boundary layer. Therefore, the density can be assumed to be a function of temperature only, i.e., $\rho = \rho(T)$. The viscosity μ also can be assumed to a function of temperature only, i.e., $\mu = \mu(T)$. Finally, the Prandtl number (Pr) is assumed nearly constant for most gases over a wide range of temperature.

DERIVATION OF GENERALIZED SIMILARITY TRANSFORMATION (ILLINGWORTH-LEVY OR LEVY-LEE TRANSFORMATION)

The derivation of a generalized similarity transformation is from the procedure adopted by Li and Nagamatsu [1] and is well summarized in [2].

ENERGY EQUATION IN TERMS OF ENTHALPY

The energy equation can be rewritten in terms of the total enthalpy.

$$H = h + \frac{u^2}{2} \tag{13}$$

where,
u : the velocity along the streamline

Using equation (13), the energy equation (3) becomes:

$$\rho u \frac{\partial H}{\partial x} + v \frac{\partial H}{\partial y} - u\left(\rho u \frac{\partial u}{\partial x} + \rho v \frac{\partial u}{\partial y} \right) =$$
$$u \frac{\partial P}{\partial x} + \mu \left(\frac{\partial u}{\partial y} \right)^2 + \frac{1}{\Pr} \left[\frac{\partial}{\partial y} \left(\mu \frac{\partial H}{\partial y} \right) - \frac{\partial}{\partial y} \left(u \mu \frac{\partial u}{\partial y} \right) \right]$$
(14)

The pressure gradient term in the energy equation (14) can be eliminated by multiplying the momentum equation (2) by u and adding the result to the energy equation (14). This results in:

$$\rho u \frac{\partial H}{\partial x} + v \frac{\partial H}{\partial y} = \left(1 - \frac{1}{\Pr} \right) \left[\mu \left(\frac{\partial u}{\partial y} \right)^2 + \frac{\partial}{\partial y} \left(u \mu \frac{\partial u}{\partial y} \right) \right]$$
$$+ \frac{1}{\Pr} \frac{\partial}{\partial y} \left(\mu \frac{\partial H}{\partial y} \right)$$
(15)

EQUATIONS IN TERMS OF STREAM FUNCTION

For the similarity transformations and the corresponding similar solutions, the compressible stream function can be defined by:

$$\frac{\partial \psi}{\partial y} = \rho u \tag{16}$$

$$\frac{\partial \psi}{\partial x} = -\rho v \tag{17}$$

Equation (16) and (17) automatically satisfy the continuity equation (1). Then, the momentum equation (2) and energy equation (15) become:

$$\begin{aligned} &\frac{\partial \psi}{\partial y} \frac{\partial}{\partial x} \left(\frac{1}{\rho} \frac{\partial \psi}{\partial y} \right) - \frac{\partial \psi}{\partial x} \frac{\partial}{\partial y} \left(\frac{1}{\rho} \frac{\partial \psi}{\partial y} \right) \\ &= -\frac{\partial P}{\partial x} + \frac{\partial}{\partial y} \left[\mu \frac{\partial}{\partial y} \left(\frac{1}{\rho} \frac{\partial \psi}{\partial y} \right) \right] \end{aligned} \tag{18}$$

$$\frac{\partial\psi}{\partial y}\frac{\partial H}{\partial x}-\frac{\partial\psi}{\partial x}\frac{\partial H}{\partial y}$$
$$=\left(1-\frac{1}{\Pr}\right)\left\{\mu\left[\frac{\partial}{\partial y}\left(\frac{1}{\rho}\frac{\partial\psi}{\partial y}\right)\right]^2+\frac{1}{\rho}\frac{\partial\psi}{\partial y}\frac{\partial}{\partial y}\left[\mu\frac{\partial}{\partial y}\left(\frac{1}{\rho}\frac{\partial\psi}{\partial y}\right)\right]\right\}$$
$$+\frac{1}{\Pr}\frac{\partial}{\partial y}\left(\mu\frac{\partial H}{\partial y}\right)^2$$
(19)

VARIABLE TRANSFORMATION

Dependent variable transformation

From the experience with the incompressible boundary layer equations, the dependent variable transformations are introduced as follows:

$$\psi(x,y)=N(x)f(\xi,\eta) \tag{20}$$
$$u(x,y)=U_e(x)f_\eta(\xi,\eta) \tag{21}$$
$$H(x,y)=H_e(x)g(\xi,\eta) \tag{22}$$

where, the subscript η indicates partial differentiation.

The form of the enthalpy transformation (22) states that the compressible boundary layer is expected to be similar with respect to a non-dimensional total enthalpy profile rather than the static enthalpy or temperature profile, as in the case for the incompressible constant-property boundary layer.

Independent variable transformation

Independent variable transformations are introduced as follows:

$$\xi=\xi(x) \tag{23}$$
$$\eta=\eta(x,y) \tag{24}$$

Relation between independent and dependent variable through trnasformation

From the definitions of the stream functions in (16) and (17):

$$\frac{\partial\psi}{\partial y}=N(x)\frac{\partial\eta}{\partial y}f_\eta(\xi,\eta)=\rho u=\rho U_e(x)f_\eta(\xi,\eta) \tag{25}$$

which results in:

$$\frac{\partial\eta}{\partial y}=\frac{U_e(x)}{N(x)}\rho \tag{26}$$

or integrating:

$$\eta=\frac{U_e(x)}{N(x)}\int_0^y\rho dy \tag{27}$$

$N(x)$ will be determined from the transformed momentum and energy equations.

FIRST FORM OF TRANSFORMED EQUATIONS

Introducing equations (23), (24), and (27) into the momentum equation (18) and energy equation (19) results in:

$$\frac{U^2}{N^2}(\rho\mu f_{\eta\eta})_\eta+\frac{N_x}{N}Uff_{\eta\eta}-U_x(f_\eta)^2-\frac{1}{\rho U}\frac{dP}{dx}$$
$$=U\xi_x(f_\eta f_{\xi\eta}-f_{\eta\eta}f_\xi) \tag{28}$$

$$\frac{U}{N^2}H_e(\rho\mu g_\eta)_\eta+\frac{N_x}{N}H_e\Pr fg_\eta-\Pr H_{e,x}f_\eta g+$$
$$(\Pr-1)\frac{U^3}{N^2}(\rho\mu f_\eta f_{\eta\eta})_\eta=H_e\xi_x(f_\eta g_\xi-f_\xi g_\eta) \tag{29}$$

where, the subscripts η , ξ , and x indicate partial differentiations.

SIMPLIFIED FORM OF TRANSFORMED EQUATIONS

Chapman - Rubesin viscosity assumption

In the equations (28) and (29), the density and the viscosity, always appear in the form $\rho\mu$ except in the pressure gradient term. This leads to the assumption of a Chapman-Rubesin viscosity law, with $w=1$ in equation (A.3) as shown in Appendix 1. Using the conditions at the edge of the boundary layer as reference condition results in:

$$\frac{\mu}{\mu_e}=C\frac{T}{T_e},(\omega=1) \tag{30}$$

which, from $\frac{\partial P}{\partial y}=0$ and the equation of state:

$$\rho\mu=C\rho_e\mu_e \tag{31}$$

Substituting this result (31) into equations (28) and (29) results in:

$$\frac{U^2}{N^2}\rho_e\mu_e(Cf_{\eta\eta})_\eta+\frac{N_x}{N}Uff_{\eta\eta}-U_x(f_\eta)^2-\frac{1}{\rho U}\frac{dP}{dx}$$
$$=U\xi_x(f_\eta f_{\xi\eta}-f_{\eta\eta}f_\xi)$$
(32)

$$\frac{U}{N^2}H_e\rho_e\mu_e(Cg_\eta)_\eta+\frac{N_x}{N}H_e\Pr fg_\eta-\Pr H_{e,x}f_\eta g+$$
$$(\Pr-1)\frac{U^3}{N^2}\rho_e\mu_e(Cf_\eta f_{\eta\eta})_\eta=H_e\xi_x(f_\eta g_\xi-f_\xi g_\eta)$$
(33)

Linear viscosity law assumption, similar assumption, and iso-energetic assumption

The coefficient $C(\eta)$ in equation (31) can vary through the boundary layer. However, the constant C assumption is made, and is evaluated at the surface conditions, e.g., using the Sutherland viscosity law in Appendix 1.

The flow is assumed to be similar, in other words, $f=f(\eta)$ and $g=g(\eta)$ such that the right hand side of the momentum equation (32) and energy equation (33) become zero.

Finally, it is assumed that the total enthalpy at the boundary layer edge is constant, i.e., $H_{e,x}=0$. This iso-energetic assumption of the inviscid flow at the edge of the boundary layer, i.e., $H_e(x)$ =constant, is not restrictive. Since $H_e=h_e+\frac{U_e^2}{2}$, both the static enthalpy and the velocity can vary along the edge of the boundary layer. From the fact that the stagnation enthalpy is constant across a shock wave, the iso-energetic flow assumption is reasonable when the shock wave is not significantly curved

From these assumptions, equation (32) becomes following (34) by replacing pressure gradient term using Euler's equation at the edge of the boundary layer, i.e., $U_x=-(1/\rho_e U_e)(dP/dx)$:

$$f'''+\frac{NN_x}{C\rho_e\mu_e U_e}ff''+\frac{N^2U_x}{C\rho_e\mu_e U_e^2}\left(\frac{\rho_e}{\rho}-f'^2\right)=0 \quad (34)$$

and, the equation (33) becomes:

$$g''+\frac{NN_x}{C\rho_e\mu_e U_e}\Pr fg'=(1-\Pr)\frac{U_e^2}{H_e}(ff'')' \quad (35)$$

where, the prime denotes ordinary differentiation with respect to η.

SIMILARITY CONDITIONS

From equations (34) and (35), the similarity conditions are:

Condition (1)

$$\frac{NN_x}{C\rho_e\mu_e U_e}=const \quad (36)$$

Condition (2)

$$\frac{N^2U_x}{C\rho_e\mu_e U_e^2}\left(\frac{\rho_e}{\rho}-f'^2\right)=\text{function of }\eta\text{ only} \quad (37)$$

Condition (3)

$$\frac{U_e^2}{H_e}=const\ \text{ or }\ \Pr=1 \quad (38)$$

Simplification from Condition (1)

If the constant in **condition (1)** is the unity, then, in the absence of a pressure gradient, the momentum equation (34) reduces to the Blasius equation in Appendix.2. In addition, comparing equation (35) with the energy equation (A.91) for forced convection in Appendix.5, by choosing the constant in **condition (1)** as unity, the differential equation (35) for the compressible boundary layer with unit Prandtl number has the same form as that for the incompressible boundary layer with an isothermal wall. Therefore, the constant in **condition (1)** is chosen as unity as follows:

$$\frac{NN_x}{C\rho_e\mu_e U_e}=1 \quad (39)$$

Rearranging and integrating of equation (39) results in:

$$N(x)=\sqrt{2\int_0^x C\rho_e\mu_e U_e dx} \quad (40)$$

Using equation (40), equation (27) becomes:

$$\eta=\frac{U_e}{\sqrt{2\int_0^x C\rho_e\mu_e U_e dx}}\int_0^y \rho dy \quad (41)$$

Since, $\xi=\xi(x)$ and:

$$\xi = \int_0^x C\rho_e\mu_e U_e dx \tag{42}$$

results in:

$$\eta = \frac{U_e(x)}{\sqrt{2\xi}}\int_0^y \rho dy \tag{43}$$

The transformations given in (42) and (43) are called the Illingworth-Levy transformation.

Simplification from condition (2)

For the case of $H_e = const.$, using the definition of $g(\eta)$ in equation (22) and the definition of the stagnation enthalpy:

$$g(\eta) = \frac{H}{H_e} = \frac{h + \frac{u^2}{2}}{h_e + \frac{U_e^2}{2}} = \frac{\left(\frac{h}{h_e} + \frac{u^2}{2h_e}\right)}{\left(1 + \frac{u^2}{2h_e}\right)} \tag{44}$$

Since, $u/U_e = f'$, equation (44) can be written as follows:

$$\frac{h}{h_e} = \left(1 + \frac{U_e^2}{2h_e}\right)g - \frac{U_e^2}{2h_e}f'^2 \tag{45}$$

Finally, from the constant pressure assumption across the boundary layer,

$$\frac{\rho_e}{\rho} - f'^2 = \frac{h}{h_e} - f'^2 = \left(1 + \frac{U_e^2}{2h_e}\right)(g - f'^2) = \frac{H_e}{h_e}(g - f'^2) \tag{46}$$

then, the term in **condition (2)** becomes:

$$\frac{N^2 U_x}{C\rho_e\mu_e U_e^2}\frac{H_e}{h_e}(g - f'^2) = \hat{\beta}(g - f'^2) \tag{47}$$

Simplification from condition (3)

Condition (3) can be written as follows:

$$\frac{U_e^2}{H_e} = \frac{U_e^2}{h_e + \frac{U_e^2}{2}} = \frac{(\gamma - 1)M_e^2}{1 + \frac{\gamma - 1}{2}M_e^2}$$

FINAL FORM OF TRANSFORMED EQUATIONS

From the above simplifications, the final governing equations are:

$$f''' + ff' + \hat{\beta}(g - f'^2) = 0 \tag{48}$$

$$g'' + \Pr fg' = \overline{\sigma}(1 - \Pr)(f'f'')' \tag{49}$$

where,

$$\overline{\sigma} = \frac{(\gamma - 1)M_e^2}{1 + \frac{\gamma - 1}{2}M_e^2}$$

ASSUMPTION FOR THE EXISTENCE OF SIMILAR SOLUTION

Power law variation in the Mach number

Similar solutions of the equations (48) and (49) exist if

$$\hat{\beta} = \frac{2\xi}{U_e}\frac{dU_e}{d\xi}\frac{H_e}{h_e} = \frac{2\xi}{U_e}\frac{dU_e}{d\xi}\left(1 + \frac{\gamma - 1}{2}M_e^2\right) \tag{50}$$

is constant.

$\hat{\beta}$ can be written in terms of the external Mach number by differentiating $[(\gamma - 1)/2]M_e^2 = U_e^2/2h_e$ and evaluating $dh_e/d\xi$ using the fact that the stagnation enthalpy is constant at the edge of the boundary layer.

$$(\gamma - 1)M_e\frac{dM_e}{d\xi} = \frac{U_e}{h_e}\frac{dU_e}{d\xi} - \frac{U_e^2}{2h_e^2}\frac{dh_e}{d\xi} = (\gamma - 1)M_e^2\left(\frac{1}{U_e}\frac{dU_e}{d\xi}\frac{H_e}{h_e}\right) \tag{51}$$

and

$$\frac{1}{U_e}\frac{dU_e}{d\xi}\frac{H_e}{h_e} = \frac{1}{M_e}\frac{dM_e}{d\xi} \tag{52}$$

From equations (51) and (52), equation (50) becomes:

$$\hat{\beta} = \frac{2\xi}{M_e}\frac{dM_e}{d\xi} = 2\left(\frac{d\ln M_e}{d\ln\xi}\right) = \text{constant} \tag{53}$$

Integrating the equation (53) results in

$$M_e = (const)\xi^{\frac{\hat{\beta}}{2}} \tag{54}$$

Therefore, the similarity requirement for the momentum equation (48) is satisfied by a power law variation of the

Mach number in the transformed plane. In addition, the similarity requirement is satisfied by an exponential Mach number variation, which is shown by Li and Nagamatsu [1] and Cohen [2].

Other assumptions

Furthermore, in order for similarity conditions in equations (48) and (49) to exist, one of the following assumptions must also be satisfied:

(1) $\gamma = 1$

(2) $M_e = 0$

(3) $\Pr = 1$

(4) $M_e = const$

(5) $\bar{\sigma} = 2$, i.e., $M_e \to \infty$

The assumption (1), i.e., $\gamma = 1$ is unrealistic for most gases.

The assumption (2), i.e., $M_e = 0$ neglects both the viscous dissipation and the compressive work terms in the energy equation. If it is further assumed that there is no heat transfer at the surface, the $M_e = 0$ assumption states that the static temperature through the boundary layer is constant. However, since the static temperature in the boundary layer should vary from the surface temperature to the static temperature at the boundary layer edge, the $M_e = 0$ assumption is less realistic than the unit Prandtl number assumption.

The assumption (3), i.e., $\Pr = 1$ states that the stagnation enthalpy or temperature for zero heat transfer at the surface is constant through the boundary layer. This result is close to the true adiabatic wall stagnation enthalpy variation, which is slight.

The assumption (4), i.e., $M_e = const$ corresponds only to the flat plate ($\hat{\beta} = 0$) in equation (54). However, for small values of the pressure gradient parameter, $\hat{\beta}$, the constant external Mach number assumption is sufficient.

The assumption (5), i.e., $M_e \to \infty$ leads to the hypersonic flow assumption, i.e., $\bar{\sigma} = 2$. This approximation is less than five percent in error at the external Mach number of ten. In addition, it allows the investigation of the effects of constant but non-unit Prandtl number on the heat transfer at the surface.

BOUNDARY CONDITIONS

Boundary conditions at the surface

The boundary conditions required at the surface for similar solutions to exist are:

$$f(0) = -\frac{\sqrt{2\xi}\,v(x)}{\mu_w U_e(x)} = const \qquad f'(0) = 0 \qquad \text{(55a, b)}$$

The equation (55a) represents the mass transfer normal to the surface. This equation is obtained by differentiating the stream function, equation (20), with respect to x, using the definition of $N(x)$ given by the equation (40) to evaluate N_x and the Chapman-Rubesin viscosity law to rearrange the result. In addition, for similar solutions to exist, $f(0)$ must be constant. This implies that:

$$v(x) = 0 \quad \text{or} \quad v(x) = \frac{\mu_w U_e(x)}{\sqrt{2\xi}} \qquad \text{(56a, b)}$$

Here, as for the Falkner-Skan equation in Appendix 3, negative values of $f(0)$ correspond to mass transfer from the surface to the fluid, i.e., injection or blowing, and positive values of $f(0)$ correspond to mass transfer from the fluid into the surface, i.e., suction.

The boundary condition related to the energy equation is:

$$g(0) = g_w = const \ \text{ or } \ g'(0) = 0 \qquad \text{(57a, b)}$$

Outer boundary conditions

The outer boundary conditions are

$$f'(\eta \to \infty) \to 1 \quad g(\eta \to \infty) \to 1 \qquad \text{(58a, b)}$$

SOLVING EQUATIONS

Similar solutions for equations (48) and (49) subject these boundary conditions (55)~(58) can be obtained according to the following cases. The results are summarized in Appendix 6. In all the bellow cases, the fundamental equations are transformed to equations similar to the fundamental equations governing the incompressible boundary layer.

Case 1: Low Speed Compressible Boundary Layer with Variable Properties

$$\hat{\beta} = const, \mathrm{M_e} = 0, \Pr = \mathrm{const}$$

$M_e = 0$ means neglecting the viscous dissipation and compressive work in the energy equation and is acceptable when the right hand side of equation (49) is small compared to the left hand side of equation (49).

The boundary value problem for this case can be written as follows:

$$f''' + ff'' + \hat{\beta}(g - f'^2) = 0 \quad (59)$$

$$g'' + \Pr fg' = 0 \quad (60)$$

with boundary conditions

$$f(0) = f_w = const \qquad f'(0) = 0 \quad (61)$$

$$g(0) = g_w = const \text{ or } g'(0) = 0 \quad (62)$$

$$f'(\eta \to \infty) \to 1 \quad g(\eta \to \infty) \to 1 \quad (63)$$

Since,

$$\frac{h(\eta)}{h_e} = g(\eta) + \frac{\gamma - 1}{2} M_e^2 (g - f'^2) \quad (64)$$

From equation (64), $M_e = 0$ case results in

$$\frac{h(\eta)}{h_e} = g(\eta)$$

Solution $g(\eta)$ of equation (60) represents non-dimensional static enthalpy profiles through the boundary layer or non-dimensional temperature profile for a constant specific heat at constant pressure (c_p).

Further, from equation (50)

$$\hat{\beta} = (2\xi / U_e)(dU_e / d\xi)$$

and, from equation (42)

$$d\xi = C\rho_e \mu_e U_e dx$$

Therefore, integration yields

$$U_e = (const)(2\xi)^{\frac{\hat{\beta}}{2}} = (const)x^{\frac{\hat{\beta}}{2-\hat{\beta}}} = (const)x^m \quad (65)$$

where,

$m \equiv \hat{\beta}/(2 - \hat{\beta})$

$\hat{\beta}$: same as β, i.e., Falkner-Skan pressure gradient parameter in Appendix 3.

Case1.1 Coupled-equations case (Nonzero heat transfer and $\hat{\beta} \neq 0$)

Due to the variable properties included in the solution, the momentum and energy equations are coupled. When there is heat transfer at the surface, the given equations in this case has no known analytical solution. This boundary value problem needs numerical methods.

Case 1.2 Uncoupled-equations ($\hat{\beta} = 0, f(0) = 0$: Flat plate without mass transfer at the surface)

For this case, momentum equation (59) reduces to Blasius equation in Appendix 2. Further, energy equation (60) has the same functional form as the energy equation governing the incompressible constant property forced convection thermal boundary layer without viscous dissipation, i.e., equation (A.62) in Appendix 4. In particular, the solution of equation (A.62) for arbitrary but constant Prandtl number is

$$g(\eta) = 1 - (1 - g_w)\theta_1(\eta) \quad (66)$$

where,

$\theta_1(\eta)$: non-dimensional solution given by equation (A.66) in Appendix 4

Case 1.3 Another uncoupled-equations - Adiabatic wall

For an adiabatic wall, i.e., $g'(0) = 0$ integrating equation (60) twice and using the boundary condition $g'(0) = 0, g(\eta \to \infty) \to 1$ results in $g(\eta) = 1$.

For zero Mach number, the static enthalpy is constant through the boundary layer. Equation (62) is analogous to the Busemann and Crocco integrals, which are, however, restricted to Pr=1. Using Equation (62), the momentum equation becomes

$$f''' + ff'' + \hat{\beta}(1 - f'^2) = 0 \quad (67)$$

This equation (67) is the Falkner-Skan equation in Appendix 2.

Although the non-dimensional momentum and energy equations are mathematically uncoupled, physically they are still coupled through the transport properties. Considering the independent variable transformation for η, from equations (43) and (46), the physical dimension y becomes:

$$\begin{aligned} y &= \int_0^\eta \frac{\left(2\int_0^x C\rho_e\mu_e U_e\right)^{1/2}}{\rho U_e} d\eta = \frac{\sqrt{2\xi}}{\rho_e U_e}\int_0^\eta \frac{\rho_e}{\rho} d\eta \\ &= \frac{\sqrt{2\xi}}{\rho_e U_e}\int_0^\eta \left[1 - (1 - f'^2) + \frac{H_e}{h_e}(g - f'^2)\right] d\eta \quad (68) \\ &= \frac{\sqrt{2\xi}}{\rho_e U_e}\left(\eta - \bar{J}_2 + \frac{H_e}{h_e}\bar{J}_1\right) \end{aligned}$$

Where,

$$\bar{J}_1 = \int_0^{\eta} (g - f'^2) d\eta$$
$$\bar{J}_2 = \int_0^{\eta} (1 - f'^2) d\eta$$

Case 2: Compressible Boundary Layer on a Flat Plate

$$\hat{\beta} = 0, \mathrm{M_e} = const, \Pr = \mathrm{const}$$

In the external inviscid flow, U_e, ρ_e and T_e are constant, equation (59) reduces to the Blasius equation in Appendix 2.

$$f''' + ff'' = 0 \tag{69}$$

The governing differential equations (69) and (49) are uncoupled. Since, the governing equations are uncoupled, they are integrated sequentially in a manner similar to that used for the incompressible constant property forced convection boundary layer in Appendix 4. And, the boundary conditions are again given by equations (61), (62), and (63).

Case 2.1 Pr=1 and Adiabatic wall

Equation (49) reduces to

$$g'' + fg' = 0 \tag{70}$$

A specific integral of this form of the energy equation (62) for zero heat transfer at the surface, i.e., adiabatic wall is $g(\eta) = 1$. However, since here $M_e \neq 0$, $g(\eta)$ is the ratio of stagnation enthalpies instead of the ratio of static enthalpies. Therefore, in this case, the stagnation enthalpy is constant through the boundary layer. Further, the adiabatic wall condition is $g_{aw} = g(0) = 1$. This means that for the unit Prandtl number, the stagnation enthalpy at the surface is equal to the stagnation enthalpy at the edge of the boundary layer. Since, the velocity is zero at the surface, for constant specific heat, the adiabatic wall temperature is equal to the stagnation temperature of the fluid at the boundary layer edge, which means, consequently, the unity recovery factor. This physical meaning is that the conversion of kinetic energy into thermal energy at the surface through the viscous dissipation is as efficient as the conversion of kinetic energy into thermal energy through the action of pressure forces in the inviscid flow at the boundary layer edge. This particular integral of the energy equation is called the Busemann energy integral.

Case 2.2 Pr=1 and Isothermal wall

$$g_w = const$$

From the Blasius equation $f''' + ff'' = 0$, if $(f' + A)$ is multiplied to the Blasius equation, where A is some constant, and add that result to the energy equation $g'' + fg' = 0$ in equation (67), the result becomes:

$$(g'' + Af''') + f(g' + Af'') + f'(f''' + ff'') = 0 \tag{71}$$

$$\rightarrow (g' + Af')'' + f(g + Af')' = 0 \tag{72}$$

Integrating once:

$$(g + Af')' = (const)e^{-\int_0^{\eta} f d\eta} \tag{73}$$

Using the boundary condition in equations (61), (62), and (63) at the surface ($\eta = 0$) yields $g'_w = (const)$ =0 for the isothermal surface.

After integrating equation (73) again and using equations (61), (62), and (63):

$$g + Af' = (const) = g_w \tag{74}$$

The constant A is evaluated from the boundary condition at infinity, equation (63).
Finally,

$$g + (g_w - 1)f' = g_w \tag{75}$$

Equation (75) is called the Crocco integral. $g_{aw} = 1$ and $g(\eta) = 1$ is a solution of the energy equation (75) for unit Prandtl number.

Case 2.3($\Pr \neq 1$)

The energy equation $g'' + \Pr fg' = \overline{\sigma}(1 - \Pr)(f'f'')'$ is a linear non-homogeneous second order ordinary differential equation with variable coefficients. The non-homogeneous term is a known forcing-function that is physically attributed to heat addition due to viscous dissipation. Since, the governing equation is linear, a solution is obtained as the sum of a complementary solution of the homogeneous equation and a particular solution of the non-homogeneous equation.

$$g = \overline{K} + KG_1 + \overline{\sigma}G_2 \tag{76}$$

where, $G_1(\eta)$ is the solution of the homogeneous boundary value problem, i.e. with original boundary conditions:

$$G_1'' + \Pr fG_1' = 0 \tag{77}$$
$$G_1(0) = 1 \quad G_1(\eta \to \infty) \to 0 \tag{78}$$

where, $G_2(\eta)$ is the solution of the non-homogeneous boundary value problem with homogeneous boundary conditions:

$$G_2'' + \Pr fG_2' = (1-\Pr)(ff'')' \quad (79)$$

$$G_2'(0) = 0 \quad G_2(\eta \to \infty) \to 0 \quad (80)$$

From the comparison of equations (A.62) & (A.63) in Appendix 4 with equations (77) & (78), $G_1(\eta) = \theta_1(\eta)$. Therefore, $G_1(\eta)$ is given by equation (A.66) in Appendix 4.

Comparing equations (A.64)&(A.65) in Appendix 4 with equations (79)&(80) reveals that the non-homogeneous terms are different. By using the method of variation of a parameter or an integrating factor,

$$G_2(\eta) = \frac{f'^2}{2} + \Pr \int_{\xi=\eta}^{\infty} (f''(\xi))^{\Pr} \left[\int_0^{\xi} (f''(\tau))^{2-\Pr} d\tau \right] d\xi$$
$$= \frac{1}{2}(f'^2 - 1) + \theta_2(\eta)$$
(81)

where,
$\theta_2(\eta)$: non-dimensional solution given by equation (A.71).
The constants K and $\overline{K}$ in equation (76) are evaluated using the boundary conditions equations (61), (62), and (63).

The complete solution is therefore

$$g(\eta) = 1 - (1-g_w)G_1(\eta) + \overline{\sigma}(G_2(\eta) - G_2(0)G_1(\eta))$$
$$= 1 - (1-g_w)\theta_1(\eta) + \overline{\sigma}(\theta_2(\eta) - \theta_2(0)\theta_1(\eta))$$
$$+ \frac{\overline{\sigma}}{2}(f'^2 + \theta_1(\eta))$$
(82)

When $\overline{\sigma} = 0$, equation (82) reduces to equation (66) obtained for $M_e = 0$.

From $\overline{\sigma} = (\gamma-1)M_e^2 / \left[1 + ((\gamma-1)/2)M_e^2\right]$, the effects of viscous dissipation on the enthalpy profile are significant when the external Mach number is significant as shown in Figure A.4 in Appendix 6. As $\overline{\sigma}$ increases the maximum enthalpy ratio in the boundary layer increases. This is a result of the conversion of kinetic energy within the boundary layer into thermal energy through viscous dissipation.

By differentiating the equation (82) and setting the result to be zero, adiabatic wall temperature can be obtained as follows.

$$g'(0) = 0 = -\theta_1'(0)\left[(1-g_{aw}) + \overline{\sigma}(\theta_2(0) - 1/2)\right] \quad (83)$$

or

$$g_{aw} = 1 + \overline{\sigma}(\theta_2(0) - 1/2) \quad (84)$$

Case 3: General Similar Compressible Boundary Layer with Unit Prandtl Number

$\hat{\beta} = const, \mathrm{M_e} = (const)\xi^{\hat{\beta}/2}, \Pr = 1$

$$f''' + ff'' + \hat{\beta}(g - f'^2) = 0 \quad (85)$$

$$g'' + fg' = 0 \quad (86)$$

with boundary conditions

$$f(0) = f_w = const \quad f'(0) = 0 \quad (87)$$

$$g(0) = g_w = const \text{ or } g'(0) = 0 \quad (88)$$

$$f'(\eta \to \infty) \to 1 \quad g(\eta \to \infty) \to 1 \quad (89)$$

This is the same non-dimensional boundary value problem governing the low speed ($M_e = 0$) compressible boundary layer, i.e., case 1. However, here the non-dimensional dependent variable g is the ratio of stagnation enthalpies H/H_e rather than h/h_e. This means that in this case 3, the effects of viscous dissipation are included.

Case 3.1 Adiabatic wall

For zero heat transfer at the surface, an explicit integral of equation (60) subject to equations (61), (62), and (63) is $g(\eta) = 1$. This shows that, for an adiabatic wall, the stagnation enthalpy is constant through the boundary layer. Therefore, for zero heat transfer at the surface, the internal heat generated due to viscous dissipation in the velocity field and the heat transferred by diffusion and conduction in the temperature field interact in a precise manner to maintain the stagnation enthalpy constant throughout the boundary layer. This result is a consequence of the unit Prandtl number assumption. As $g(\eta) = 1$, the non-dimensional momentum equation reduces to the Falkner-Skan equation in Appendix 3. Therefore, the nondimensional momentum and energy equations are uncoupled.

Case 3.2 Non-zero heat transfer at the surface case

When there is heat transfer at the surface, the given equations in this case has no known analytical solution. This boundary value problem was studied numerically by

Cohen [2], Levy [3], Li & Nagamatsu [1], Cohen & Reshotko [4], and Rogers [5].

Case 4: Similar Hypersonic Compressible Boundary Layer

$$\hat{\beta} = const, \mathrm{M_e} \to \infty, \Pr \neq 1$$

$$f''' + ff'' + \hat{\beta}(g - f'^2) = 0 \tag{90}$$

$$g'' + \Pr fg' = 2(1-\Pr)(ff'')' \tag{91}$$

with boundary conditions

$$f(0) = f_w = const \qquad f'(0) = 0 \tag{92}$$

$$g(0) = g_w = const \text{ or } g'(0) = 0 \tag{93}$$

$$f'(\eta \to \infty) \to 1 \quad g(\eta \to \infty) \to 1 \tag{94}$$

For the cases 1 to 3, under certain conditions, the boundary value problem for the compressible boundary layer could be reduced to an equivalent incompressible boundary layer problem. However, this is not possible for the present case 4. This is because the stagnation enthalpy is not constant through the boundary layer even for an adiabatic wall ($g(\eta) = 1$ is not an integral of the energy equation). Thus, since, $\hat{\beta} \neq 0$, the momentum equation (90) cannot be reduced to the Falkner-Skan equation in Appendix 3. Therefore, the functions $f(\eta), f'(\eta)$ and $f''(\eta)$ required in the energy integrals, equations (A.66) and (81), depend on $g(\eta), g'(\eta)$ and $g''(\eta)$ because of the coupling between the momentum and energy equations, (59) and (91). Because of this coupling, the energy integrals cannot be evaluated except by successive approximations using the incompressible Falkner-Skan solutions to begin the approximation. Therefore, numerical method should be used to get the exact solution.

RELIABILITY OF THE SIMILARITY TRANSFORMATION METHODS

Experimental data presented in Figure A.6 in Appendix 6 suggest that the proposed transformations predict the velocity and enthalpy of the system with high accuracy (e.g. velocity profile for the compressible boundary layer on an adiabatic flat plate)

It should be noted, however, that real life applications are most likely to deviate from one of these four categories presented above. The need for numerical simulation is then becoming essential for more accuracy. However, the analytical approach is critical as it provides the essential framework on which the numerical approximations are built.

OTHER SIMILARITY TRANSFORMATIONS

HOWARTH TRANSFORMATION

Howarth transformation here is a restricted form of the transformation similar to the one due to Howarth [7] and the following derivation is adopted from ([2][8]).

Introducing the compressible stream function defined by

$$\frac{\partial \psi}{\partial y} = \frac{\rho u}{\rho_r} \quad \text{and} \quad \frac{\partial \psi}{\partial x} = -\frac{\rho v}{\rho_r} \tag{95}$$

where, the subscript r indicates some reference condition.

Independent variable transformations are:

$$\xi = \xi(x) \qquad \eta = \eta(x, y) \tag{96}$$

, which are subject to the condition $u = \psi_\eta$.

The particular functional forms chosen for the independent variable transformations are based on the equivalent forms for incompressible flow with restriction $u = \psi_\eta$ which is also based on the incompressible results.

$$\psi_\eta = \frac{\rho_r}{\rho} \quad \psi_y = \frac{\partial y}{\partial \eta}\psi_y \quad \frac{\partial \eta}{\partial y} = \frac{\rho}{\rho_r} \tag{97}$$

which, yields the required independent variable transformation for η, i.e.

$$\eta = \int_0^y \frac{\rho}{\rho_\infty} dy \tag{98}$$

The formal transformation equations are

$$\frac{\partial}{\partial y} = \frac{\rho}{\rho_r}\frac{\partial}{\partial \eta} \tag{99}$$

$$\frac{\partial}{\partial x} = \xi_x \frac{\partial}{\partial \xi} + \eta_x \frac{\partial}{\partial \eta} \tag{100}$$

Transform of the momentum equation

Using equations (99) and (100), the transformed momentum equation becomes:

$$\xi_x(\psi_\eta \psi_{\xi\eta} - \psi_\xi \psi_{\eta\eta}) = -\frac{1}{\rho}\frac{\partial P}{\partial x} + \frac{1}{\rho_r^2}(\rho\mu\psi_{\eta\eta})_\eta \tag{101}$$

Using the Chapman-Rubesin viscosity law with w=1 in Appendix 1, i.e.

$$\frac{\mu}{\mu_r} = C\frac{T}{T_r} \tag{102}$$

and from $\partial P/\partial y = 0$, and using the equation of state,

$$\rho\mu = C\rho_r\mu_r \tag{103}$$

then, equation (101) becomes

$$\xi_x(\psi_\eta\psi_{\xi\eta} - \psi_\xi\psi_{\eta\eta}) = -\frac{1}{\rho}\frac{\partial P}{\partial x} + \nu_r(C\psi_{\eta\eta})_\eta \tag{104}$$

If C =constant, then with $\xi_x = C$, i.e., $\xi = Cx$,

$$(\psi_\eta\psi_{\xi\eta} - \psi_\xi\psi_{\eta\eta}) = -\frac{1}{C}\frac{\rho_r}{\rho}\frac{1}{\rho_r}\frac{\partial P}{\partial x} + \nu_r\psi_{\eta\eta\eta} \tag{105}$$

Except for the factor $(1/C)(\rho_r/\rho)$ in the pressure gradient term, equation (105) has the same form as the momentum equation governing incompressible constant property boundary layer flow. When the pressure gradient is zero, i.e., for a flat plate at zero-incidence, equation (105) has exactly the same form as the incompressible constant property momentum equation. In the absence of pressure gradient, the similarity transformations developed for the incompressible boundary layer flow yield the Blasius equation in Appendix 2, i.e.

$$\eta^* = \eta\left(\frac{U(\xi)}{2\nu\xi}\right)^{1/2} \qquad \psi(\xi,\eta) = \left(2\nu U(\xi)\xi\right)^{1/2} f^*(\eta^*)$$
(106)

$$(f^*)''' + f^*(f^*)'' = 0 \tag{107}$$

The transformed boundary conditions for an impermeable surface are

$$f^*(0) = f^*(0)' = 0 \tag{108}$$

$$f^{*'}(\eta^* \to \infty) \to 1 \tag{109}$$

where, the prime denotes differentiation with respect to η^*.

Therefore, the solution of the momentum equation for the compressible variable property boundary layer in the absence of pressure gradient is reduced to the solution of an equivalent incompressible constant property equation, i.e., the Blasius equation in Appendix 2. In the absence of a pressure gradient, the momentum equation for compressible boundary layer flow is uncoupled from the energy equation. Formally this is true. However, determining the physical coordinate, y, from the inverse of equation (98), requires a knowledge of the density distribution in the boundary layer and therefore the solution of the energy equation. Thus, the momentum and energy equations for compressible boundary layer flow, even in the absence of a pressure gradient, are still technically coupled.

<u>Transform of the energy equation</u>

Transformation of the energy equation into ξ,η coordinates yields

$$\psi_\eta h_\xi - \psi_\xi h_\eta = \frac{1}{C}\frac{\rho_r}{\rho}\psi_\eta\frac{1}{\rho_r}\frac{\partial P}{\partial x} + \frac{1}{C\,\mathrm{Pr}\,\rho_r^2}(\rho\mu h_\eta)_\eta$$
$$+\frac{\rho\mu}{C}\frac{1}{\rho_r^2}(\psi_{\eta\eta})^2$$
(110)

Introducing the Chapman-Rubesin viscosity law in Appendix 1 and assuming that $C\rho_r\mu_r$ is constant,

$$\psi_\eta h_\xi - \psi_\xi h_\eta = \frac{1}{C}\frac{\rho_r}{\rho}\psi_\eta\frac{1}{\rho_r}\frac{\partial P}{\partial x} + \frac{\nu_r}{\mathrm{Pr}}h_{\eta\eta} + \nu_r(\psi_{\eta\eta})^2$$
(111)

In the absence of a pressure gradient, equation (105) is equivalent to the energy equation governing forced convection flow over a flat plate at zero-incidence.

STEWARTSON-ILLINGWORTH TRANSFORMATION

Assuming that unity Prnadtl number, constant c_p, and the viscosity linearly related to the temperature, Stewartson and Illingworth have independently shown that there exists a transformation from a compressible flow boundary layer, to a related incompressible flow boundary layer ([4][8]).

A stream function that satisfies the continuity equation is:

$$\frac{\partial\psi}{\partial y} = \frac{\rho u}{\rho_0} \tag{112}$$

$$\frac{\partial\psi}{\partial x} = -\frac{\rho v}{\rho_0} \tag{113}$$

The energy and momentum equations are transformed to new coordinates X and Y such that:

$$X = \int_0^x C \frac{P_e}{P_0} \frac{a_e}{a_0} dx \quad (114)$$

$$Y = \int_0^y C \frac{\rho_e}{\rho_0} dy \quad (115)$$

where, a means sonic speed and subscript 0 represents some reference state.

The enthalpy function S is defined as:

$$S = \frac{h_e}{h_0} - 1 \quad (116)$$

The stream function is replaced by the transformed velocities U and V through following relations.

$$U = \frac{\partial \psi}{\partial Y} \quad (117)$$

$$V = -\frac{\partial \psi}{\partial X} \quad (118)$$

Equations (112)-(118) are applied to the momentum and energy equations and a new set of equations is obtained. It assumed that the pressure is constant along the boundary layer and that wall temperature is constant.

In order to reduce this system into a system of ordinary differential equation, the following relations are assumed:

$$\psi = A X^a U_e^p f(\eta) \quad (119)$$

$$Y = B X^b U_e^q \eta \quad (120)$$

$$S = S(\eta) \quad (121)$$

where, A, B, a, b, p, and, q are undetermined variables.

Possible similar solutions are possible if:

$U_e = CX^m$ or $U_e = C \exp(C_2 X)$ (122)

Then, the system of ODEs corresponding to the power-law velocity distribution of equations my be written:

$$f''' + ff'' = \beta(f'^2 - 1 - S) \quad (123)$$

$$S'' + \Pr fS' = (1 - \Pr) \frac{(\gamma - 1) M_e^2}{1 + \frac{\gamma - 1}{2} M_e^2} (ff''' + f''^2) \quad (124)$$

where,

$\beta = \frac{2m}{m+1}$: Pressure gradient

$\frac{U}{U_e} = \frac{u}{u_e} = f'$: Velocity ratio

SUMMARY

With the increased complexity of the equations of motion for compressible (variable-density), variable-property flows, it was natural to seek ways of rigorously extending the material at hand for constant-density, constant-property flows to those cases. Ways were sought to transform a compressible boundary layer problem into an equivalent incompressible problem. The existing solutions could then be transformed back to a solution for the original compressible problem. This procedure ended in success with some assumptions. We discussed three examples, e.g. the Illingworth-Levy transformation, the Howarth transformation and the Stewartson-Illingworth transformation.

REFERENCES

[1] Li, T.Y., and Nagamatsu, H.T., Similar solutions of compressible boundary layer over a flat plate with suction or injection, JAS, Vol.22, pp.589-597, 1955

[2] David F. Rogers, Laminar flow analysis, Cambridge University Press, 1992

[3] Cohen, C., Similar solutions of compressible laminar boundary layer equations, JAS, Vol.21, pp.281-282, 1949

[4] Levy, S., Effect of large temperature changes (including viscous heating) upon laminar boundary layers with variable free-stream velocity, JAS, Vol.21, pp.341-348, 1952

[5] Cohen, C.B., and Reshotko, E., Similar solutions for the compressible laminar boundary layer with heat transfer and pressure gradient, NACA TR 1293, 1956

[6] Rogers, D.F, Reverse flow solutions for compressible laminar boundary layer equations, Phys. of Fluids, Vol. 12, pp. 517-523, 1969

[7] Howarth, L., Concerning the effect compressibility on laminar boundary layers and their separation, Proc. Roy. Soc. London Ser. A, Vol. 194, pp. 16-42, 1948

[8] Stefan Schreier, Compressible flow, Wiley, 1982

[9] A.D. Young, Boundary layers, Oxford, 1989

OTHER SOURCES

[1] O.A. Oleinik, V.N. Samokin, Mathematical models in boundary layer theory, Chapman&Hall/CRC, 1999

[2] Josep A. Schetz, Boundary layer analysis, Prentice-Hall, 1993

[3] Herrmann Schlichting, Klaus Gerstan, Boundary-layer theory, Springer, 2000

APPENDICES

APPENDIX 1. VARIATION OF TRANSPORT PROPERTIES

The transport properties of importance in a viscous compressible flow are the viscosity, the thermal conductivity, the specific heat at constant pressure, and the Prandtl number which is the combination of the first three properties.

VISCOSITY

From monatomic gas theory, the viscosity of gases depends only on the temperature and is independent of the pressure. Experimental measurements confirm that this result is essentially correct for all gases. For gases, the viscosity increases with increasing temperature. In contrast, the viscosity of liquids depends on both temperature and pressure and decreases with increasing temperature.

<u>Sutherland viscosity law</u>

Experimental measurements of the viscosity of air are related with temperature by the Sutherland equation:

$$\frac{\mu}{\mu_r} = \left(\frac{T}{T_r}\right)^{3/2} \frac{T_r + S_1}{T + S_1} \qquad \text{(A.1)}$$

For air between 180 $^\circ R$ and 3400 $^\circ R$,

S_1 =198.6 $^\circ R$

T_r =491.6 $^\circ R$

$\mu_r = 3.58 \times 10^{-7} (lb\sec/ft^2)$

Therefore, for air,

$$\mu = 2.270 \frac{T^{3/2}}{T + 198.6} \times 10^{-8} (lb\sec/ft^2) \qquad \text{(A.2)}$$

Figures A.1 and A.2 show the absolute viscosity of certain gases and liquids and the power law viscosity relationship respectively [2].

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Figure A.1. Absolute viscosity of certain gases and liquids

Image removed due to copyright considerations.

Figure A.2. Power law viscosity relationship

<u>Chapman-Rubesin viscosity law</u>

Because of the complexity of Sutherland equation, approximation formula based on the empirical equation called Chapman-Rebesin viscosity law is used instead.

$$\frac{\mu}{\mu_r} = C\left(\frac{T}{T_r}\right)^{\omega} \qquad \text{(A.3)}$$

A simple and useful case of the Chapman-Rubesin viscosity law occurs when C=1 and w=1 in equation (A.3). With these values, and using the surface as the reference condition,

$$\mu = \mu_w(x)\frac{T}{T_w(x)}$$

For an isothermal wall, this reduces to

$\mu = (const)T$

THERMAL CONDUCTIVITY

The thermal conductivity of gases k also depends only on the temperature and is independent of pressure. The variation of the thermal conductivity of air with temperature is the same as that of the dynamic viscosity.

SPECIFIC HEAT AT CONSTANT PRESSURE

The specific heat at constant pressure c_p for air is almost constant for a wide range of temperatures.

THE PRANDTL NUMBER

The behaviors of transport properties with temperature mentioned above make the Prandtl number $\Pr = \mu c_p / k$ essentially invariant with temperature. Therefore, it is assumed that the Prandtl number for gases is constant. This assumption eliminates the need to formally specify the functional variation of the c_p and k with temperature. Further, considerable mathematical simplification occurs if we choose a unit Prandtl number and a Chapman-Rubesin viscosity law with C=w=1.

Figure A.3 shows the variation of k, cp and Pr with temperature [2].

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Figure A.3. Variation of thermal conductivity, specific heat at constant pressure, and the Prandtl number with temperature

APPENDIX 2. BLASIUS EQUATION- THE FLOW PAST A FLAT PLATE WITHOUT PRESSURE GRADIENT

GOVERNING EQUATIONS AND BOUDNARY CONDTION

Continuity equation

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{A.4}$$

Momentum equation

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2} \tag{A.5}$$

Boundary condition

$$y = 0: \ u = v = 0 \tag{A.6}$$

$$y \to \infty: \ u \to U \tag{A.7}$$

TRANSORMATION USING STREAM FUNCTION

$$u = \psi_y \qquad v = -\psi_x \tag{A.8}$$

From (A.5) and (A.8),

$$\psi_y \psi_{xy} - \psi_x \psi_{yy} = \nu \psi_{yyy} \tag{A.9}$$

With boundary conditions

$$y = 0 \qquad \psi_x = \psi_y = 0 \tag{A.10}$$

$$y \to \infty \quad \psi_y \to U \tag{A.11}$$

After similarity transformation

$$\eta = y\sqrt{\frac{U}{2\nu x}} \qquad \psi(x, y) = \sqrt{2\nu U x}\, f(\eta) \tag{A.12}$$

$$f''' + ff'' = 0 \tag{A.13}$$

with $f(0) = f'(0) = 0$ (A.14)

and $f'(\eta \to \infty) \to 1$ (A.15)

APPENDIX 3. FALKNER-SKAN EQUATION - THE FLOW PAST A FLAT PLATE WITH PRESSURE GRADIENT

GOVERNING EQUATIONS AND BOUDNARY CONDTION

Continuity equation

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 \tag{A.16}$$

Momentum equation

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=-\frac{1}{\rho}\frac{\partial P}{\partial x}+v\frac{\partial^2 u}{\partial y^2} \tag{A.17}$$

Boundary condition

$$y=0:\ u=v=0 \tag{A.18}$$

$$y\to\infty : u\to U(x) \tag{A.19}$$

TRANSORMATION USING STREAM FUNCTION

$$\psi_y\psi_{xy}-\psi_x\psi_{yy}=U\frac{dU}{dx}+v\psi_{yyy} \tag{A.20}$$

With boundary conditions

$$y=0 \quad \psi_x=\psi_y=0 \tag{A.21}$$

$$y\to\infty \quad \psi_y\to U(x) \tag{A.22}$$

By introducing the transformations

$$\xi=x \qquad \eta=\frac{Ay}{g(x)} \tag{A.23}$$

$$\psi(x,y)=BU(x)g(x)f(\xi,\eta) \tag{A.24}$$

Then the governing equations yield:

$$f'''+\frac{(AB)^2}{vA^3B}g(Ug)'ff''+\frac{g^2U'}{vA^3B}[1-(AB)^2f'^2] \\ =\frac{(AB)^2}{vA^3B}g^2U(ff'_\xi-f''f_\xi) \tag{A.25}$$

where,

$$f'=\frac{\partial f}{\partial \eta} \quad f_\xi=\frac{\partial f}{\partial \xi} \quad f'_\xi=\frac{\partial f'}{\partial \xi}$$

$$U'=\frac{dU}{dx} \quad g'=\frac{dg}{dx}$$

For all $x>0$, the boundary condition at infinity ($y\to\infty,\eta\to\infty$) becomes:

$$\psi_y=ABU(x)f'(\xi,\eta\to\infty)\to U(x) \tag{A.26}$$

or $f'(\xi,\eta\to\infty)\to\frac{1}{AB}$ (A.27)

In order to non-dimensionalize equation (A.25) and to obtain a simple numerical result for the boundary condition at infinity, AB=1 and $vA^3B=U_\infty$ are chosen.

Where, U_∞: the potential velocity upstream of $x=0$.

Solving for A and B yields

$$A=\sqrt{U_\infty/v} \quad \text{and} \quad B=\sqrt{v/U_\infty} \tag{A.28}$$

Then, equation (A.25) becomes:

$$f'''+\alpha ff''+\beta(1-f'^2)=g^2\frac{U}{U_\infty}(ff'_\xi-f''f_\xi) \tag{A.29}$$

where,

$$\alpha=\frac{g}{U_\infty}(Ug)' \quad \beta=\frac{g^2}{U_\infty}U' \tag{A.30}$$

In order for similar solutions to exist, the transformed stream function must be a function of η only, i.e., $f=f(\eta)$. Therefore, the right hand side of equation (A.29) must be zero. Furthermore, α,β must be independent of x. Since g and U were assumed to be functions of x only, α,β are constants.

$$f'''+\alpha ff''+\beta(1-f'^2)=0 \tag{A.31}$$

with boundary conditions

$$f(0)=f'(0)=0 \tag{A.32}$$

$$f(\eta\to\infty)\to 1 \tag{A.33}$$

Since α,β are assumed to be constants, equations (A.30) represent two equations in the two unknown functions, U(x) and g(x). U(x) and g(x) can thus be determined. From

$$2\alpha-\beta=\frac{2g}{U_\infty}(Ug)'-\frac{g^2}{U_\infty}U'=\frac{1}{U_\infty}(g^2U)' \tag{A.34}$$

Providing $2\alpha - \beta \neq 0$, integration of equation (A.34) yields

$$\frac{U}{U_\infty} g^2 = (2\alpha - \beta)x \tag{A.35}$$

A second algebraic equation for U(x) and g(x) is obtained by considering

$$\alpha - \beta = \frac{g}{U_\infty}(Ug)' - \frac{g^2}{U_\infty}U' = \frac{U}{U_\infty} gg' \tag{A.36}$$

Multiplying both sides of equation (A.36) by U' and rewriting results in:

$$(\alpha - \beta)\frac{U'}{U} = \frac{g^2}{U_\infty}U'\frac{g'}{g} = \beta\frac{g'}{g} \tag{A.37}$$

Integration of equation (A.37) results in:

$$\left(\frac{U}{U_\infty}\right)^{\alpha-\beta} = U_\infty^{\beta-\alpha} g^\beta = Kg^\beta \tag{A.38}$$

Simultaneous solution of equations (A.34) and (A.38) yields

$$\frac{U(x)}{U_\infty} = K^{\frac{2}{2\alpha-\beta}}[(2\alpha - \beta)x]^{\frac{\beta}{2\alpha-\beta}} = (const)x^m \tag{A.39}$$

and

$$g(x) = \left[(2\alpha - \beta)\frac{U_\infty x}{U}\right]^{1/2} \tag{A.40}$$

Similar solution of the steady two-dimensional incompressible boundary layer exist if the potential velocity U(x) varies as a power of the distance along the surface. Providing $\alpha \neq 0$, without loss of generality, $\alpha = 1$ is chosen. In addition, by introducing

$$m = \frac{\beta}{2 - \beta} \quad \text{or} \quad \beta = \frac{2m}{m+1} \tag{A.41}$$

U(x) and g(x) then become

$$U(x) = \overline{K}x^m \quad \text{and} \quad g(x) = \left(\frac{2}{m+1}\frac{x}{U}U_\infty\right)^{1/2} \tag{A.42}$$

where,

$$\overline{K} = U_\infty K^{m+1}\left(\frac{2}{m+1}\right)^m \tag{A.43}$$

Using the results for A and g(x) in the original transformations, yields the appropriate independent similarity variable

$$\eta = y\sqrt{\frac{(m+1)}{2}\frac{U}{\nu x}} \tag{A.44}$$

In the above analysis, the cases where $\alpha = 0$ and where $2\alpha - \beta = 0$ were excluded.

APPENDIX 4. FORCED CONVECTION BOUNDARY LAYER WITHOOUT PRESSURE GRADIENT– PARALLE FLOW PAST A FLAT PLATE

GOVERNING EQUATIONS AND BOUNDARY CONDITION

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Figure A.4. Forced convection boundary layer flow past a flat plate [2]

Continuity equation

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{A.45}$$

Momentum equation

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2} \tag{A.46}$$

Energy equation

$$\rho c_p\left(u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y}\right) = k\frac{\partial^2 T}{\partial y^2} + \mu\left(\frac{\partial u}{\partial y}\right)^2 \tag{A.47}$$

Boundary condition

$$y = 0 \quad u = v = 0 \tag{A.48}$$

$$T = T_w \quad \text{or} \quad \frac{\partial T}{\partial y} = 0 \tag{A.49}$$

$$y \to \infty \quad u \to U_\infty \quad T \to T_\infty \tag{A.50}$$

where, the subscript ∞ indicates conditions in the inviscid flow at the edge of the boundary layer.

SIMILARITY TRANSFORMATION

Using the similarity transformations

$$\eta = y\sqrt{\frac{U}{2\nu x}} \qquad \psi(x,y) = \sqrt{2\nu U x}\, f(\eta) \quad \text{(A.51)}$$

$$f''' + ff'' = 0 \tag{A.52}$$

with $f(0) = f'(0) = 0$ (A.53)

and $f'(\eta \to \infty) \to 1$ (A.54)

$$\frac{d^2T}{d\eta^2} + \Pr f\frac{dT}{d\eta} = -\Pr\frac{U_\infty^2}{c_p} f''^2 \tag{A.55}$$

Introducing the non-dimensional temperature

$$\theta = \frac{T - T_\infty}{T_w - T_\infty} \tag{A.56}$$

Then, equation (A.55) becomes

$$\theta'' + \Pr f\theta' = -\Pr E f''^2 \tag{A.57}$$

with $\theta(0) = 1$ or $\theta'(0) = 0$ (A.58)

and $\theta(\eta \to \infty) \to 0$ (A.59)

where, the appropriate Eckert number is defined as

$$E = \frac{U_\infty^2}{c_p(T_w - T_\infty)} \tag{A.60}$$

Equation (A.57) is a second-order linear non-homogeneous differential equation subject to two-point asymptotic boundary conditions. This equation can be divided by two equations by superposition principle.

$$\theta(\eta) = K\theta_1(\eta) + E\theta_2(\eta) \tag{A.61}$$

Homogeneous equation

$$\theta_1'' + \Pr f\theta_1' = 0 \tag{A.62}$$

with $\theta_1(0) = 1$ and $\theta_1(\eta \to \infty) \to 0$ (A.63)

Non-homogeneous equation

$$\theta_2'' + \Pr f\theta_2' = -\Pr f''^2 \tag{A.64}$$

with $\theta_2'(0) = 0$ and $\theta_2(\eta \to \infty) \to 0$ (A.65)

SOLUTION (ANALYTICAL SOLUTION)

Both the homogeneous and the non-homogeneous boundary value problems are amenable to analytical solution.

Homogeneous solution

The solution of the homogeneous problem is

$$\theta_1(\eta) = \frac{\int_{\xi=\eta}^{\infty} (f''(\xi))^{\Pr} d\xi}{\int_{\xi=0}^{\infty} (f''(\xi))^{\Pr} d\xi} = \alpha_0(\Pr)\int_{\xi=\eta}^{\infty} (f''(\xi))^{\Pr} d\xi$$

(A.66)

where, $\alpha_0(\Pr) = \int_{\xi=0}^{\infty} (f''(\xi))^{\Pr} d\xi$

For the special case of unit Prandtl number $\alpha_0(1) = 1$, equation (A.66) reduces to

$$\theta_1(\eta) = 1 - f'(\eta) \tag{A.67}$$

Therefore, when there is heat transfer at the surface the non-dimensional temperature distribution has the same form as the non-dimensional velocity distribution.

Non-homogeneous solution

The non-homogeneous boundary value problem for the adiabatic wall is solved using the method of variation of a parameter.

$$\theta_2(\eta) = \Pr\int_{\xi=\eta}^{\infty} (f''(\xi))^{\Pr}\left(\int_0^{\xi} (f''(\tau))^{2-\Pr} d\tau\right) d\xi$$

(A.68)

For the special case of unit Prandtl number, this result reduces to

$$\theta_2(\eta) = \frac{1}{2}(1 - f'^2(\eta)) \tag{A.69}$$

Total solution

Using the condition $\theta = 1$ at $\eta = 0$

$$K = 1 - E\theta_2(0) \tag{A.70}$$

and equation (A.61) becomes

$$\theta(\eta) = [1 - E\theta_2(0)]\theta_1(\eta) + E\theta_2(\eta) \quad \text{(A.71)}$$

For the special case of unit Prandtl number, $\theta_2(0) = \dfrac{1}{2}$

$$\theta(\eta) = \theta_1(\eta) + E\left(\theta_2(\eta) - \frac{\theta_1(\eta)}{2}\right) \quad \text{(A.72)}$$

APPENDIX 5. FORCED CONVECTION BOUNDARY LAYER WITH PRESSURE GRADIENT AND NONISOTHERMAL SURFACE CONDITION

GOVERNING EQUATIONS AND BOUNDARY CONDITION

Continuity equation

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad \text{(A.73)}$$

Momentum equation

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{dP}{dx} + \nu\frac{\partial^2 u}{\partial y^2} \quad \text{(A.74)}$$

Energy equation

$$u\frac{\partial h}{\partial x} + v\frac{\partial h}{\partial y} = \frac{\nu}{\Pr}\frac{\partial^2 h}{\partial y^2} + \nu\left(\frac{\partial u}{\partial y}\right)^2 \quad \text{(A.75)}$$

Boundary condition

$$y = 0 \quad u = v = 0 \quad \text{(A.76)}$$

$$T = T(x) \quad \text{(A.77)}$$

$$y \to \infty \quad u \to U_\infty \quad T \to T_e \quad \text{(A.78)}$$

SIMILARITY CONDITION FOR THE VELOCITY FIELD (FALKNER-SKAN EQUATION)

$$U(x) = U_\infty x^m \qquad m = \frac{\beta}{2-\beta} \quad \text{(A.79)}$$

SIMILARITY TRANSFORAMTION

$$\eta = y\sqrt{\frac{(m+1)}{2}\frac{U}{\nu x}} \quad \text{(A.80)}$$

$$\psi(x,y) = \left[\frac{2}{m+1}\nu x U\right]^{1/2} f(\eta) \quad \text{(A.81)}$$

TRANSFORMED EQUATION

$$f''' + ff'' + \beta(1 - f'^2) = 0 \quad \text{(A.82)}$$

$$\begin{aligned} &f(0) = f'(0) = 0 \\ &f'(\eta \to \infty) \to 1 \end{aligned} \quad \text{(A.83)}$$

$$\frac{\partial^2 T}{\partial \eta^2} + \Pr f\frac{\partial T}{\partial \eta} + 2\Pr\frac{m-1}{m+1}\eta f'\frac{\partial T}{\partial \eta} - \frac{2\Pr}{m+1}f'x\frac{\partial T}{\partial x} = -\Pr\frac{U^2}{c_p}f''^2$$

(A.84)

or

$$\frac{\partial^2 T}{\partial \eta^2} + \Pr f\frac{\partial T}{\partial \eta} + 2\Pr(\beta-1)\eta f'\frac{\partial T}{\partial \eta} - (2-\beta)\Pr f'x\frac{\partial T}{\partial x} = -\Pr\frac{U^2}{c_p}f''^2$$

(A.85)

By introducing a non-dimensional temperature:

$$\theta = \frac{T - T_w}{T_e - T_w} \quad \text{(A.86)}$$

Further assumption is made such that the surface heat transfer is sufficiently small that it does not change the temperature in the inviscid flow at the edge of the boundary layer. Thus, $T_e = T_\infty$

$$\theta'' + \Pr f\theta' - \Pr(2-\beta)(1-\theta)f'\frac{x}{T_e - T_w}\frac{dT_w}{dx} = -\Pr\frac{U_\infty^2}{c_p(T_e - T_w)}x^{2m}f''^2 \quad \text{(A.87)}$$

CONDITION FOR THE EXISTENCE OF SIMILARITY SOLUTION

$$\frac{x}{T_e - T_w}\frac{dT_w}{dx} = const \quad \text{and} \quad \frac{x^{2m}}{T_e - T_w} = const \quad \text{(A.88)}$$

$$\rightarrow T_w - T_e = T_1 x^n \quad \text{(A.89)}$$

where,
T_1 : a constant associated with the initial temperature distribution (n=0 : Isothermal condition)

Using this assumption, equation (A.87) becomes

$$\theta'' + \Pr f\theta' + n\Pr(2-\beta)(1-\theta)f' = \Pr E_1 x^{2m-n} f''^2$$

where,
the Eckert number is $E = U_\infty^2 / c_p T_1$

The associated boundary conditions are

$$\theta(0) = 0 \qquad \theta(\eta \to \infty) = 1 \tag{A.90}$$

SOLUTION

From equation (A.89), there are two classes of similar solutions of the energy equation for forced convection: those with viscous dissipation, and those without viscous dissipation

<u>Low speed incompressible flow (Neglect of viscous dissipation)</u>

In this case, the Eckert number is small, since U_∞ is small. Under these conditions, the viscous dissipation on the right hand side can be neglected.

$$\theta'' + \Pr f\theta' + n\Pr(2-\beta)(1-\theta)f' = 0 \tag{A.91}$$

1) n = 0 (isothermal wall)

This equation reduces to the same form as the homogeneous solution for the flat plate isothermal wall case, i.e., equation (A.62). Although equation (A.91) is of the same form as equation (A.62), its solution $\theta = \theta(\eta)$ is not the same. Here, the non-dimensional stream function $f(\eta)$, given by the solution of the Falkner-Skan equation in Appendix 3, depends on the value of β, and, in turn, the solution of equation (A.91) also depends on the value of β.

2) $\beta = 2$

Similar solutions of the energy equation exist for arbitrary wall temperature variations.

3) $n \neq 0$ and $\beta \neq 2$

The similar solutions of the energy equation depend on both the pressure gradient β and the surface temperature parameter n.

<u>When the viscous dissipation is not neglected</u>

Similar solutions of the energy equation exits only if

2m-n=0.→ $n = 2\beta/(2-\beta)$ and $\beta \neq 2$

In other words, similar solutions of the energy equation exist for only one wall temperature variation.

1) $0 < \beta < 2$
→The surface temperature increases in the direction of the flow

2) $\beta < 0$
→ The surface temperature decreases in the direction of the flow

3) $\beta = 0$ and $n = 0$
→ Constant surface temperature

APPENDIX 6. SUMMARY OF GOVERNING EQUATIONS FOR SIMILAR COMPRESSIBLE BOUNDARY LAYER

<u>Case 1: The low speed compressible boundary layer</u>

$\hat{\beta} \neq 0$ and $M_e = 0$

<u>Equations</u>

$$f''' + ff'' + \hat{\beta}(g - f'^2) = 0$$
$$g'' + \Pr fg' = 0$$

<u>1.1 Adiabatic Wall analytical Solutions</u> (It can be solved using the result of Falkner-Skan equation)

$$f''' + ff'' + \hat{\beta}(1 - f'^2) = 0$$
$$g(\eta) = 1$$

<u>1.2 Isothermal Wall analytical Solutions</u> (It can not be solved)

<u>Comments:</u>
1) $g(\eta) = \dfrac{h}{h_e}$
2) The viscous dissipation terms are neglected

<u>Case 2: The compressible boundary layer on a flat plate</u>

$\hat{\beta} \neq 0$ and $M_e = const$

<u>Equations:</u>

$$f''' + ff'' = 0$$
$$g'' + fg' = \bar{\sigma}(1-\Pr)(ff'')'$$

<u>2.1 Adiabatic Wall analytical Solutions</u> (It can be solved for Pr=1):

$$f''' + ff'' = 0$$
$g(\eta) = 1$ (Busseman Integral)

2.2 Isothermal Wall analytical Solutions (It can be solved)

2.2.1 $\Pr = 1$

$$f''' + ff'' = 0$$
$$g(\eta) = g_w - (g_w - 1)f' \text{ (Crocco integral)}$$

2.2.2 $\Pr \neq 1$

$$f''' + ff'' = 0$$
$$g(\eta) = 1 - (1 - g_w)\theta_1(\eta) + \overline{\sigma}(\theta_2(\eta) - \theta_2(0)\theta_1(\eta))$$
$$+ \frac{\overline{\sigma}}{2}(f'^2 + \theta_1(\eta))$$

Comments:

1) $g(\eta) = \dfrac{H}{H_e}$

2) The viscous dissipation terms are included

3) For $\Pr = 1$, $g_{aw} = 1$

For $\Pr \neq 1$, $g_{aw} = \overline{\sigma}(\theta_2(0) - \frac{1}{2})$

Figure A.5 presents the effect of Mach number on the enthalpy profiles for a flat plate [2]:

Image removed due to copyright considerations.

Figure A.5. The effect of Mach number on the enthalpy for $\hat{\beta} = 0$, $g_w = 0.6$, and Pr=0.723

Figure A.6 shows the comparison of experimental and theoretical velocity profiles for the compressible boundary layer on an adiabatic flat plate [2]:

Image removed due to copyright considerations.

Figure A.6. Comparison of experimental and theoretical velocity profiles for the compressible boundary layer on an adiabatic flat plate

Case 3:similar compressible boundary layer with unit prandtl number

$$\hat{\beta} \neq 0, M_e = (const)\xi^{\frac{\hat{\beta}}{2}}, \Pr = 1$$

Equations:
$$f''' + ff'' + \hat{\beta}(g - f'^2) = 0$$
$$g'' + \Pr fg' = 0$$

3.1 Adiabatic Wall analytical Solutions (It can be solved):

$$f''' + ff'' + \hat{\beta}(g - f'^2) = 0$$
$$g(\eta) = 1$$

3.2 Isothermal Wall analytical Solutions (It cannot be solved)

Comments:

1) No analytical solutions for non-unit Prandtl number

2) The viscous dissipation terms are included

Case 4: The similar hypersonic compressible boundary layer with nonunit prandtl number

$$\hat{\beta} \neq 0 \quad M_e \to \infty$$

Equations:
$$f''' + ff'' + \hat{\beta}(g - f'^2) = 0$$
$$g'' + fg' = 2(1 - \Pr)(f'f'')'$$

4.1 Adiabatic Wall analytical Solutions (Cannot be solved):

4.2 Isothermal Wall analytical Solutions (Cannot be solved):

Comments:

1) $M_e \to \infty$ yields $\overline{\sigma} = 2$

2) $g(\eta) \neq 1$ for the adiabatic wall