Magnus Effect

Consider the flow past a spinning cylinder. In a real fluid, the angular motion would act to impart a net circulation to the flow through the action of the fluid viscosity. This circulation (denoted by the constant Γ) may be incorporated artificially into an irrotational flow model by adding an irrotational vortex potential $\Phi = -\frac{\theta\Gamma}{2\pi}$ to the velocity potential deduced for potential flow over a cylinder. So the total potential will be of this form:

$$\Phi = U\left(r + \frac{a^2}{r}\right)cos\theta - \frac{\theta\Gamma}{2\pi}$$

- (a) Calculate an expression for the resulting velocity field.

- (b) By examining the location of stagnation points in the flow, deduce the dependence of the form of the flow on the dimensionless spin number $S = \frac{\Gamma}{(4\pi aU)}$, and make rough sketches of the flow for $S < 1$, $S = 1$ and $S > 1$.

- (c) Demonstrate that the drag on the cylinder still vanishes regardless of the spin number.

- (d) Deduce an expression for the transverse force (or "lift") on the cylinder.

Note: the generation of lift through the interaction of circulation and translation is the root of many interesting phenomena in the dynamics of sports (generically known as the Magnus Effect).

Solution:

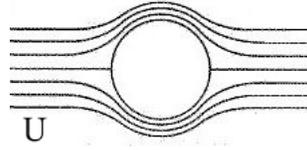(a) For a cylinder in uniform flow we have:

Figure 1: Cylinder in uniform flow with velocity U

$$w_1(z) = U\left(z + \frac{a^2}{z}\right) \tag{1}$$

Now for "simulating" the flow around a cylinder which is spinning in uniform flow, we add a clockwise line vortex of circulation $-\Gamma$:

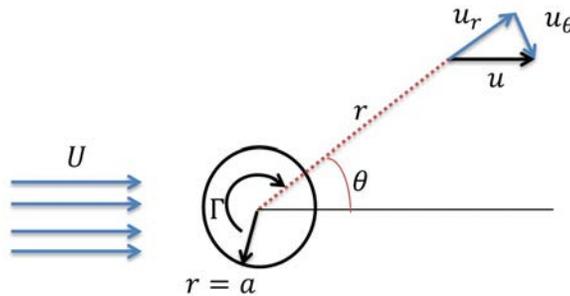$$w_2(z) = \frac{i\Gamma}{2\pi} \ln\left(\frac{z}{a}\right) \tag{2}$$

Figure 2: Cylinder in uniform flow with clockwise circulation

Therefore:

$$w(z) = w_1 + w_2 = U\left(z + \frac{a^2}{z}\right) + \frac{i\Gamma}{2\pi} \ln\left(\frac{z}{a}\right) \tag{3}$$

Knowing that $z = re^{i\theta}$:

$$w(z) = U\left(re^{i\theta} + \frac{a^2}{r}e^{-i\theta}\right) + \frac{i\Gamma}{2\pi}\left(\ln\left(\frac{r}{a}\right) + i\theta\right) = U\left[\left(r + \frac{a^2}{r}\right)\cos\theta + i\sin\theta\left(r - \frac{a^2}{r}\right)\right] + \frac{i\Gamma}{2\pi}\ln\left(\frac{r}{a}\right) - \frac{\theta\Gamma}{2\pi} \tag{4}$$

Thus, writing $w(z)$ as $w(z) = \Phi(z) + i\Psi(z)$:

$$w(z) = \Phi(z) + i\Psi(z) = U\left(r + \frac{a^2}{r}\right)\cos\theta - \frac{\theta\Gamma}{2\pi} + i\left(U\sin\theta\left(r - \frac{a^2}{r}\right) + \frac{\Gamma}{2\pi}\ln\left(\frac{r}{a}\right)\right) \tag{5}$$

$$\Phi(z) = U\left(r + \frac{a^2}{r}\right)\cos\theta - \frac{\theta\Gamma}{2\pi} \tag{6}$$

$$\Psi(z) = U\sin\theta\left(r - \frac{a^2}{r}\right) + \frac{\Gamma}{2\pi}\ln\left(\frac{r}{a}\right) \tag{7}$$

Then we can calculate the velocities in the r and θ directions:

$$u_r = \frac{\partial\Phi}{\partial r} = \frac{1}{r}\frac{\partial\Psi}{\partial\theta} = U\left(1 - \frac{a^2}{r^2}\right)\cos\theta \tag{8}$$

$$u_\theta = \frac{1}{r}\frac{\partial\Phi}{\partial\theta} = -\frac{\partial\Psi}{\partial r} = -U\left(1 + \frac{a^2}{r^2}\right)\sin\theta - \frac{\Gamma}{2\pi r} \tag{9}$$

Checking the boundary conditions, at the boundaries of the cylinder $(r = a)$ there should be no flux.

$$\rightarrow u_r(r = a) = U\left(1 - \frac{a^2}{a^2}\right)\cos\theta = 0 \tag{10}$$

Since we are in potential flow, we should not care about no-slip as the potential flow does not satisfy the no-slip boundary condition. The other boundary condition is that at $r \to \infty$ velocity goes back to U.

$$u_\theta(r \to \infty) = -U\sin\theta \tag{11}$$

$$u_r(r \to \infty) = U\cos\theta \tag{12}$$

$$|u(r \to \infty)| = \overline{u_r{}^2(r \to \infty) + u_\theta{}^2(r \to \infty)} = U \tag{13}$$

(b) In order to find the stagnation point from equations (8) and (9):

$$u_r = 0 \rightarrow \quad r = a \quad or \quad \cos(\theta) = 0 \tag{14}$$

$$u_\theta = 0 \quad and \quad r = a:$$

$$-U(2)\sin\theta - \frac{\Gamma}{2\pi a} = 0 \rightarrow \quad \sin\theta = -\frac{\Gamma}{4\pi aU} \tag{15}$$

$$\tag{16}$$

So, if $\frac{\Gamma}{4\pi aU} < 1$, then stagnation points are on the cylinder at angles:

$$\theta_1 = \sin^{-1}\left(-\frac{\Gamma}{4\pi aU}\right) \tag{17}$$

$$\theta_2 = \pi - \sin^{-1}\left(-\frac{\Gamma}{4\pi aU}\right) \tag{18}$$

If $\frac{\Gamma}{4\pi aU} = 1$, there there will be only one stagnation point on the cylinder at:

$$\theta = -\frac{\pi}{2} \tag{19}$$

And if $\frac{\Gamma}{4\pi aU} < 1$, then there is no stagnation point on the cylinder, but we can use that fact that at $\cos\theta = 0$ the radial velocity is zero (This cannot be on the cylinder,$/r \neq a$). Therefore:

$$\cos\theta = 0 \quad \rightarrow \theta = -\frac{\pi}{2} , \theta = \frac{\pi}{2} \tag{20}$$

$$\theta = \frac{\pi}{2} \rightarrow \ u_\theta = -U\left(1 + \frac{a^2}{r^2} - \frac{\Gamma}{2\pi r}\right) = 0 \quad (Impossible \ for \ \Gamma > 1) \tag{21}$$

$$\theta = -\frac{\pi}{2} \rightarrow \ u_\theta = U\left(1 + \frac{a^2}{r^2}\right) - \frac{\Gamma}{2\pi r} = 0 \tag{22}$$

$$\rightarrow \ r = \frac{1}{4\pi U}\left(\Gamma + \sqrt{\Gamma^2 - (4\pi aU)^2}\right) \tag{23}$$

Equation (22) has 2 roots, but since one root lies inside the cylinder ($r < a$), it is discarded and only the root given in equation (23) is acceptable. A sketch for the flow profiles of the 3 cases mentioned is shown in figure 3.
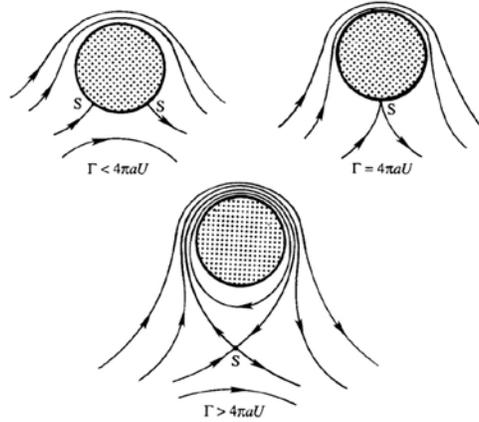

Figure 3: Different regimes of flow past a circular cylinder with circulation. Adapted from Fluid Mechanics 4th ed.,P. K. Kundu and I. M. Cohen, Academic Press, 2008

(c) To find the drag and lift forces, we need to find the pressure first; for any point on the cylinder we can use Bernoulli:

$$P_\infty + \frac{1}{2}\rho U^2 = P(r = a, \theta) + \rho u_\theta{}^2 \tag{24}$$

$$\rightarrow P(r = a, \theta) = P_\infty + \frac{1}{2}\rho\left[U^2 - \left(-2U\sin\theta - \frac{\Gamma}{2\pi a}\right)^2\right] \tag{25}$$

Equation (25) is symmetric about the y-axis ($\theta = \frac{\pi}{2}, -\frac{\pi}{2}$), therefore there is no force component in the x direction and therefore, there is no drag force on the cylinder.

$$F_D = -\int_0^{2\pi} P(r = a, \theta)\cos\theta a d\theta = -a\int_0^{2\pi}\left(P_\infty + \frac{1}{2}\rho\left[U^2 - \left(2U\sin\theta + \frac{\Gamma}{2\pi a}\right)^2\right]\right)\cos\theta d\theta = 0 \tag{26}$$

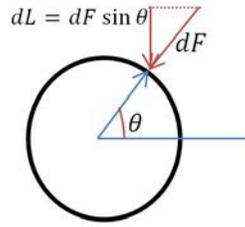(d)To find lift, again using the pressure from equation (25):

Figure 4: Free body diagram of the cylinder

$$dF = -P(r = a, \theta)ad\theta \tag{27}$$
$$dL = dF \sin\theta \tag{28}$$

$$L = -\int_0^{2\pi} P(r = a, \theta)\sin\theta a d\theta = -a\int_0^{2\pi}\left(P_\infty + \frac{1}{2}\rho\left[U^2 - \left(2U\sin\theta + \frac{\Gamma}{2\pi a}\right)^2\right]\right)\sin\theta d\theta$$

$$L = a\rho\int_0^{2\pi}\frac{\Gamma U}{\pi a}\sin^2\theta d\theta = \rho\frac{\Gamma U}{\pi}\int_0^{2\pi}\frac{1 - \cos2\theta}{2}d\theta = \rho U\Gamma$$

$$L = \rho U\Gamma \tag{29}$$

Equation (29) is true for irrotational flow around any 2D object and not only cylinders. This equations is known as the Kutta-Zhukhousky lift theorem.

$\square$

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