2.20 - Marine Hydrodynamics
Lecture 9

Lecture 9 is structured as follows: In paragraph 3.5 we return to the full Navier-Stokes equations (unsteady, viscous momentum equations) to deduce the vorticity equation and study some additional properties of vorticity. In paragraph 3.6 we introduce the concept of potential flow and velocity potential. We formulate the governing equations and boundary conditions for potential flow and finally introduce the stream function.

3.5 Vorticity Equation

Return to viscous incompressible flow. The Navier-Stokes equations in vector form

$$\frac{\partial \vec{v}}{\partial t} + \vec{v} \cdot \nabla \vec{v} = -\nabla \left(\frac{p}{\rho} + gy \right) + \nu \nabla^2 \vec{v}$$

By taking the curl of the Navier-Stokes equations we obtain the vorticity equation. In detail and taking into account $\nabla \times \vec{u} \equiv \vec{\omega}$ we have

$$\nabla \times (\text{Navier-Stokes}) \rightarrow \nabla \times \frac{\partial \vec{v}}{\partial t} + \nabla \times (\vec{v} \cdot \nabla \vec{v}) = -\nabla \times \nabla \left(\frac{p}{\rho} + gy \right) + \nabla \times \left(\nu \nabla^2 \vec{v} \right)$$

The first term on the left side, for fixed reference frames, becomes

$$\nabla \times \frac{\partial \vec{v}}{\partial t} = \frac{\partial}{\partial t} (\nabla \times \vec{v}) = \frac{\partial \vec{\omega}}{\partial t}$$

In the same manner the last term on the right side becomes

$$\nabla \times \left(\nu \nabla^2 \vec{v} \right) = \nu \nabla^2 \vec{\omega}$$

Applying the identity $\nabla \times \nabla \cdot scalar = 0$ the pressure term vanishes, provided that the density is uniform

$$\nabla \times \left(\nabla \left(\frac{p}{\rho} + gy\right) \right) = 0$$

The inertia term $\vec{v} \cdot \nabla \vec{v}$, as shown in Lecture 8, §3.4, can be rewritten as

$$\vec{v} \cdot \nabla \vec{v} = \frac{1}{2} \nabla \left(\vec{v} \cdot \vec{v} \right) - \vec{v} \times \left(\nabla \times \vec{v} \right) = \nabla \left(\frac{v^2}{2} \right) - \vec{v} \times \vec{\omega} \text{ where } v^2 \equiv \left| \vec{v} \right|^2 = \vec{v} \cdot \vec{v}$$

and then the second term on the left side can be rewritten as

$$\begin{aligned} \nabla \times \left(\vec{v} \cdot \nabla \right) \vec{v} &= \nabla \times \nabla \left(\frac{v^2}{2} \right) - \nabla \times \left(\vec{v} \times \vec{\omega} \right) = \nabla \times \left(\vec{\omega} \times \vec{v} \right) \\ &= \left(\vec{v} \cdot \nabla \right) \vec{\omega} - \left(\vec{\omega} \cdot \nabla \right) \vec{v} + \underbrace{\vec{\omega} \left(\nabla \cdot \vec{v} \right)}_{\substack{=0 \\ \text{incompressible} \\ \text{fluid}}} + \underbrace{\vec{v} \left(\nabla \cdot \vec{\omega} \right)}_{\substack{=0 \text{ since} \\ \nabla \cdot (\nabla \times \vec{v}) = 0}} \end{aligned}$$

Putting everything together, we obtain the vorticity equation

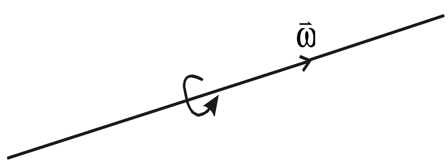
$$\boxed{\frac{D\vec{\omega}}{Dt} = \left(\vec{\omega} \cdot \nabla \right) \vec{v} + \nu \nabla^2 \vec{\omega}}$$

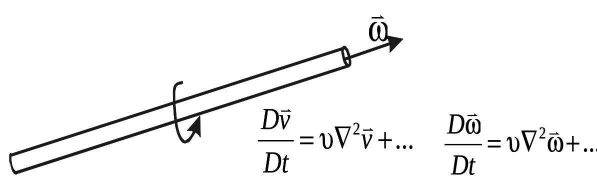
Comments-results obtained from the vorticity equation

- Kelvin's Theorem revisited - from vorticity equation:

  If $\nu \equiv 0$, then $\frac{D\vec{\omega}}{Dt} = \left(\vec{\omega} \cdot \nabla \right) \vec{v}$, so if $\vec{\omega} \equiv 0$ everywhere at one time, $\vec{\omega} \equiv 0$ always.

- ν can be thought of as diffusivity of vorticity (and momentum), i.e., $\vec{\omega}$ once generated (on boundaries only) will spread/diffuse in space if ν is present.





- Diffusion of vorticity is analogous to the heat equation: $\dfrac{\partial T}{\partial t} = K\nabla^2 T$, where K is the heat diffusivity.

  Numerical example for $\nu \sim 1$ mm^2/s. For diffusion time $t = 1$ second, diffusion distance $L \sim O\left(\sqrt{\nu t}\right) \sim O\left(mm\right)$. For diffusion distance $L = 1$cm, the necessary diffusion time is $t \sim O\left(L^2/\nu\right) \sim O(10)$sec.

- In 2D space (x, y),

$$\vec{v} = (u, v, 0) \quad \text{and} \quad \frac{\partial}{\partial z} \equiv 0$$

So, $\vec{\omega} = \nabla \times \vec{v}$ is $\perp$ to $\vec{v}$ ($\vec{\omega}$ is parallel to the z-axis). Then,

$$(\vec{\omega} \cdot \nabla)\,\vec{v} = \left(\underbrace{\omega_x}_{0}\frac{\partial}{\partial x} + \underbrace{\omega_y}_{0}\frac{\partial}{\partial y} + \omega_z \underbrace{\frac{\partial}{\partial z}}_{0}\right)\vec{v} \equiv 0,$$

so in 2D we have

$$\frac{D\vec{\omega}}{Dt} = \nu\nabla^2\vec{\omega}$$

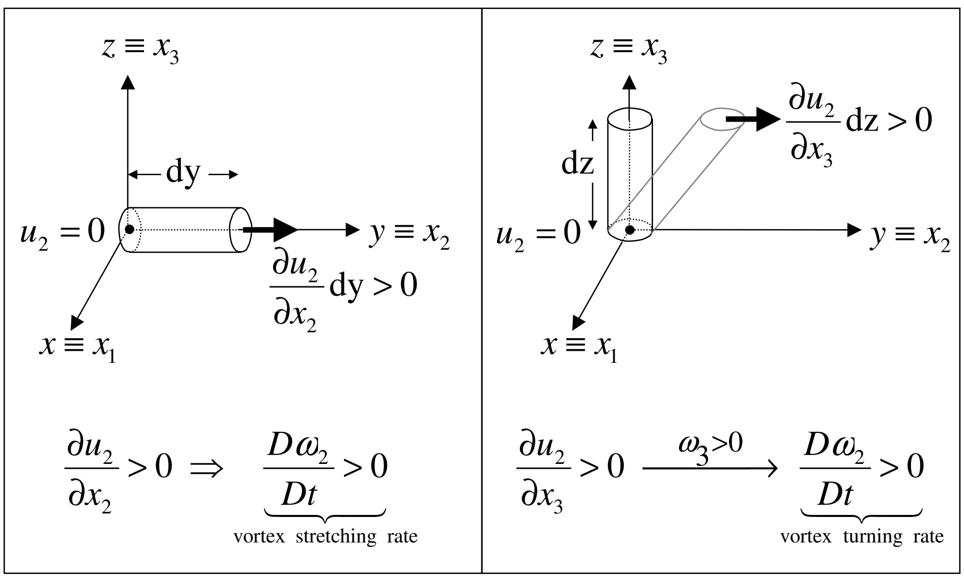
If $\nu = 0$, $\frac{D\vec{\omega}}{Dt} = 0$, i.e., in 2D following a particle the angular velocity is conserved.
Reason: In 2D space the length of a vortex tube cannot change due to continuity.

- In 3D space,

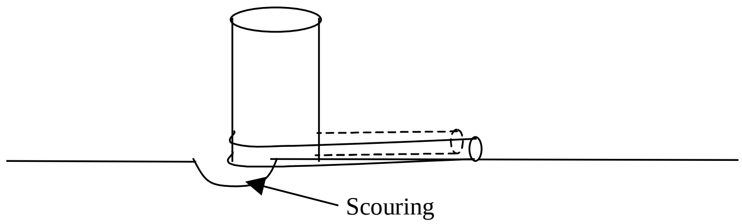
$$\frac{D\omega_i}{Dt} = \underbrace{\omega_j \frac{\partial v_i}{\partial x_j}}_{\text{vortex turning and stretching}} + \underbrace{\nu \frac{\partial^2 \omega_i}{\partial x_j \partial x_j}}_{\text{diffusion}}$$

for example,

$$\frac{D\omega_2}{Dt} = \underbrace{\omega_1 \frac{\partial u_2}{\partial x_1}}_{\text{vortex turning}} + \underbrace{\omega_2 \frac{\partial u_2}{\partial x_2}}_{\text{vortex stretching}} + \underbrace{\omega_3 \frac{\partial u_2}{\partial x_3}}_{\text{vortex turning}} + \text{diffusion}$$



3.5.1 **Example: Pile on a River**



What really happens as length of the vortex tube L increases?

IFCF is no longer a valid assumption.

Why?

Ideal flow assumption implies that the inertia forces are much larger than the viscous effects. The Reynolds number, with respect to the vortex tube diameter D is given by

$$R_e \sim \frac{UD}{\nu}$$

As the vortex tube length increases $\Rightarrow$ the diameter D becomes really small $\Rightarrow$ R_e is not that big after all.

Therefore IFCF is no longer valid.

3.6 Potential Flow

Potential Flow (**P-Flow**) is an ideal and irrotational fluid flow

$$\textbf{P-Flow} \equiv \begin{cases} \left.\begin{array}{cc} \textbf{Inviscid Fluid} & \nu = 0 \\ + & \\ \textbf{Incompressible Flow} & \nabla \cdot \vec{v} = 0 \end{array}\right\} \textbf{Ideal Flow} \\ \quad + \\ \textbf{Irrotational Flow} \quad \vec{\omega} = 0 \textbf{ or } \Gamma = 0 \end{cases}$$

3.6.1 Velocity potential

For ideal flow under conservative body forces by Kelvin's theorem if $\vec{\omega} \equiv 0$ at some time t, then $\vec{\omega} \equiv 0 \equiv$ irrotational flow always. In this case the flow is P-Flow.

Given a vector field $\vec{v}$ for which $\vec{\omega} = \nabla \times \vec{v} \equiv 0$, there exists a potential function (scalar) - the velocity potential - denoted as ϕ, for which

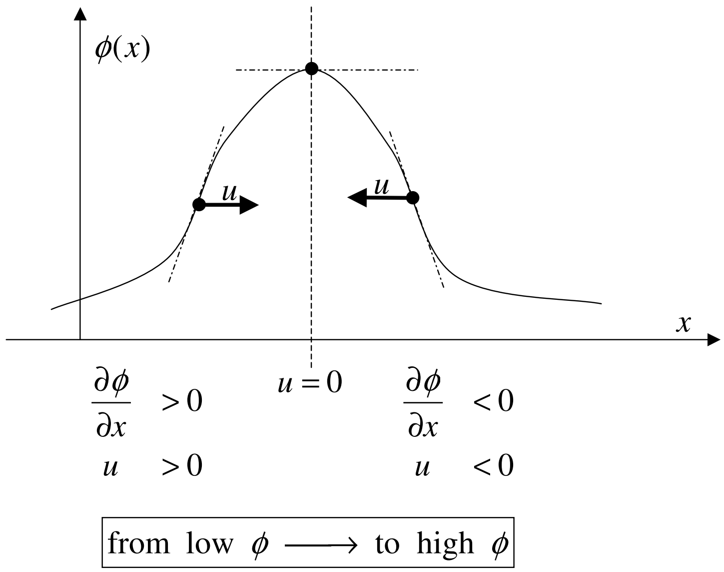
$$\vec{v} = \nabla \phi$$

Note that

$$\vec{\omega} = \nabla \times \vec{v} = \nabla \times \nabla \phi \equiv 0$$

for any ϕ, so irrotational flow guaranteed automatically. At a point $\vec{x}$ and time t, the velocity vector $\vec{v}(\vec{x}, t)$ in cartesian coordinates in terms of the potential function $\phi(\vec{x}, t)$ is given by

$$\vec{v}\left(\vec{x}, t\right) = \nabla \phi \left(\vec{x}, t\right) = \left(\frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y}, \frac{\partial \phi}{\partial z}\right)$$



The velocity vector $\vec{v}$ is the gradient of the potential function ϕ, so it always points towards higher values of the potential function.

3.6.2 Governing Equations and Boundary Conditions for Potential Flow

(a) Continuity

$$\nabla \cdot \vec{v} = 0 = \nabla \cdot \nabla\phi \Rightarrow \nabla^2\phi = 0$$

Number of unknowns $\rightarrow \quad \phi$

Number of equations $\rightarrow \quad \nabla^2\phi = 0$

Therefore we have closure. In addition, the velocity potential ϕ and the pressure p are decoupled. The velocity potential ϕ can be solved independently first, and after ϕ is obtained we can evaluate the pressure p.

$$p = f(\vec{v}) = f(\nabla\phi) \quad \rightarrow \quad \text{Solve for } \phi, \text{ then find pressure.}$$

(b) Bernoulli equation for P-Flow

This is a **scalar** equation for the pressure under the assumption of **P-Flow** for **steady** or **unsteady** flow.

Euler equation:

$$\frac{\partial \vec{v}}{\partial t} + \nabla\left(\frac{v^2}{2}\right) - \vec{v}\times\vec{\omega} = -\nabla\left(\frac{p}{\rho} + gy\right)$$

Substituting $\vec{v} = \nabla\phi$ and $\vec{\omega} = 0$ into Euler's equation above, we obtain

$$\nabla\left(\frac{\partial \phi}{\partial t}\right) + \nabla\left(\frac{1}{2}\left|\nabla\phi\right|^2\right) = -\nabla\left(\frac{p}{\rho} + gy\right)$$

or

$$\nabla\left\{\frac{\partial \phi}{\partial t} + \frac{1}{2}\left|\nabla\phi\right|^2 + \frac{p}{\rho} + gy\right\} = 0,$$

which implies that

$$\frac{\partial \phi}{\partial t} + \frac{1}{2}\left|\nabla\phi\right|^2 + \frac{p}{\rho} + gy = f(t)$$

everywhere in the fluid for unsteady, potential flow. The equation above can be written as

$$p = -\rho\left[\frac{\partial \phi}{\partial t} + \frac{1}{2}\left|\nabla\phi\right|^2 + gy\right] + F(t)$$

which is the **Bernoulli equation for unsteady or steady potential flow**.

DO NOT CONFUSE WITH
BERNOULLI EQUATION FROM § 3.4,
USED FOR STEADY, ROTATIONAL FLOW

Summary: Bernoulli equation<u>S</u> for ideal flow.

(a) For steady rotational or irrotational flow along streamline:

$$p = -\rho\left(\frac{1}{2}v^2 + gy\right) + C(\psi)$$

(b) For unsteady or steady irrotational flow everywhere in the fluid:

$$p = -\rho\left(\frac{\partial\phi}{\partial t} + \frac{1}{2}\left|\nabla\phi\right|^2 + gy\right) + F(t)$$

(c) For hydrostatics, $\vec{v} \equiv 0, \frac{\partial}{\partial t} = 0$:

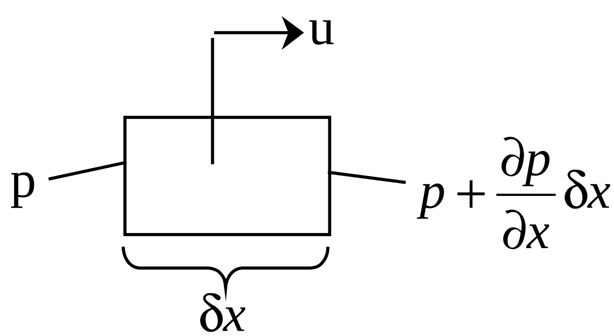
$$p = -\rho gy + c \leftarrow \text{ hydrostatic pressure (Archimedes' principle)}$$

(d) Steady and no gravity effect ($\frac{\partial}{\partial t} = 0, g \equiv 0$):

$$p = -\frac{\rho v^2}{2} + c = -\frac{\rho}{2}\left|\nabla\phi\right|^2 + c \leftarrow \text{Venturi pressure (created by velocity)}$$

(e) Inertial, acceleration effect:

$$\begin{aligned} p &\sim - \overbrace{\rho\frac{\partial\phi}{\partial t}}^{\text{Eulerian inertia}} + \cdots \\ \nabla p &\sim - \rho\frac{\partial}{\partial t}\vec{v} + \cdots \end{aligned}$$



(c) Boundary Conditions

- KBC on an impervious boundary

$$\underbrace{\vec{v}\cdot\hat{n}}_{\hat{n}\cdot\nabla\phi} = \underbrace{\vec{u}\cdot\hat{n}}_{U_n \text{ given}} \quad \text{no flux across boundary} \;\Rightarrow \frac{\partial\phi}{\partial n} = U_n \text{ given}$$

- DBC: specify pressure at the boundary, i.e.,

$$-\rho\left(\frac{\partial\phi}{\partial t} + \frac{1}{2}\left|\nabla\phi\right|^2 + gy\right) = \text{given}$$

Note: On a free-surface $p = p_{atm}$.

3.6.3 Stream function

- Continuity: $\nabla \cdot \vec{v} = 0$; Irrotationality: $\nabla \times \vec{v} = \vec{\omega} = 0$
- Velocity potential: $\vec{v} = \nabla\phi$, then $\nabla \times \vec{v} = \nabla \times (\nabla\phi) \equiv 0$ for any ϕ, i.e., irrotationality is satisfied automatically. Required for continuity:

$$\nabla \cdot \vec{v} = \nabla^2 \phi = 0$$

- Stream function $\vec{\psi}$ defined by

$$\vec{v} = \nabla \times \vec{\psi}$$

Then $\nabla \cdot \vec{v} = \nabla \cdot \left(\nabla \times \vec{\psi}\right) \equiv 0$ for any $\vec{\psi}$, i.e., satisfies continuity automatically. Required for irrotationality:

$$\nabla \times \vec{v} = 0 \Rightarrow \nabla \times \left(\nabla \times \vec{\psi}\right) = \underbrace{\nabla\left(\nabla \cdot \vec{\psi}\right) - \nabla^2\vec{\psi}}_{\substack{\text{still 3 unknown} \\ \vec{\psi}=(\psi_x,\psi_y,\psi_z)}} = 0 \tag{1}$$

- For 2D and axisymmetric flows, $\vec{\psi}$ is a scalar ψ (stream functions are more 'useful' for 2D and axisymmetric flows).

For 2D flow: $\vec{v} = (u, v, 0)$ and $\frac{\partial}{\partial z} \equiv 0$.

$$\vec{v} = \nabla \times \vec{\psi} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \psi_x & \psi_y & \psi_z \end{vmatrix} = \left(\frac{\partial}{\partial y}\psi_z\right)\hat{i} + \left(-\frac{\partial}{\partial x}\psi_z\right)\hat{j} + \left(\frac{\partial}{\partial x}\psi_y - \frac{\partial}{\partial y}\psi_x\right)\hat{k}$$

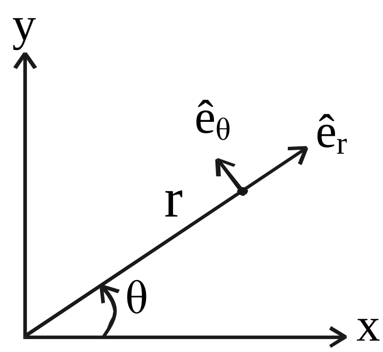
Set $\psi_x = \psi_y \equiv 0$ and $\psi_z = \psi$, then $u = \frac{\partial\psi}{\partial y}$; $v = -\frac{\partial\psi}{\partial x}$

So, for 2D:

$$\nabla \cdot \vec{\psi} = \frac{\partial}{\partial x}\psi_x + \frac{\partial}{\partial y}\psi_y + \frac{\partial}{\partial z}\psi_z \equiv 0$$

Then, from the irrotationality (see (1)) $\Rightarrow$ $\nabla^2\psi = 0$ and ψ satisfies Laplace's equation.

- 2D polar coordinates: $\vec{v} = (v_r, v_\theta)$ and $\frac{\partial}{\partial z} \equiv 0$.



$$\vec{v} = \nabla \times \vec{\psi} = \frac{1}{r} \begin{vmatrix} \hat{e}_r & r\hat{e}_\theta & \hat{e}_z \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial z} \\ \psi_r & \psi_\theta & \psi_z \end{vmatrix} = \overbrace{\frac{1}{r}\frac{\partial \psi_z}{\partial \theta}}^{v_r} \hat{e}_r \overbrace{- \frac{\partial \psi_z}{\partial r}}^{v_\theta} \hat{e}_\theta + \overbrace{\frac{1}{r}\left(\frac{\partial}{\partial r} r\psi_\theta - \frac{\partial}{\partial \theta}\psi_r \right)}^{v_z} \hat{e}_z$$

Again let

$$\psi_r = \psi_\theta \equiv 0 \text{ and } \psi_z = \psi \text{ , then}$$

$$v_r = \frac{1}{r}\frac{\partial \psi}{\partial \theta} \text{ and } v_\theta = -\frac{\partial \psi}{\partial r}$$

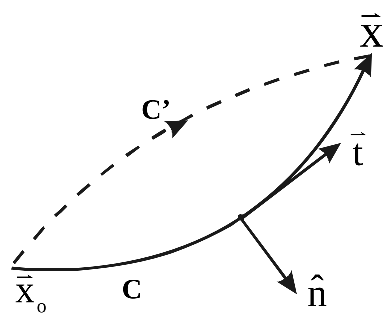
- For 3D but axisymmetric flows, $\vec{\psi}$ also reduces to ψ (read JNN 4.6 for details).

- **Physical Meaning of ψ.**
  In 2D

$$u = \frac{\partial\psi}{\partial y} \text{ and } v = -\frac{\partial\psi}{\partial x}$$

We define

$$\psi(\vec{x},t) = \psi(\vec{x}_0,t) + \underbrace{\int_{\vec{x}_0}^{\vec{x}} \vec{v}\cdot\hat{n}d\ell}_{\substack{\text{total volume flux}\\ \text{from left to right}\\ \text{accross a curve C}\\ \text{between } \vec{x} \text{ and } \vec{x}_0}} = \psi(\vec{x}_0,t) + \int_{\vec{x}_0}^{\vec{x}} (udy - vdx)$$

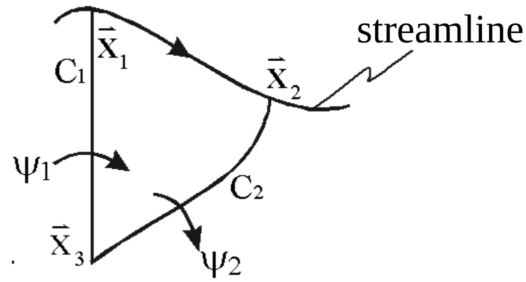


For ψ to be single-valued, $\int$ must be path independent.

$$\int_C = \int_{C'} \text{ or } \int_C - \int_{C'} = 0 \longrightarrow \oint_{C-C'} \vec{v}\cdot\hat{n}\ d\ell = \iint_S \underbrace{\nabla\cdot\vec{v}}_{=0, \text{ continuity}}\ ds = 0$$

Therefore, ψ is unique because of continuity.

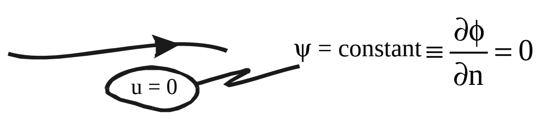
Let $\vec{x}_1, \vec{x}_2$ be two points on a given streamline ($\vec{v}\cdot\hat{n} = 0$ on streamline)

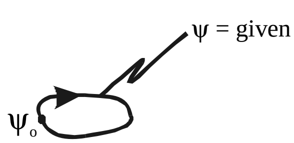


$$\underbrace{\psi\left(\vec{x}_2\right)}_{\psi_2} = \underbrace{\psi\left(\vec{x}_1\right)}_{\psi_1} + \int_{\vec{x}_1}^{\vec{x}_2} \underbrace{\vec{v}\cdot\hat{n}}_{\substack{=0 \\ \text{along a} \\ \text{streamline}}} \, d\ell$$

Therefore, $\psi_1 = \psi_2$, i.e., ψ is a constant along any streamline. For example, on an impervious stationary body $\vec{v}\cdot\hat{n} = 0$, so $\psi =$ constant on the body is the appropriate boundary condition. If the body is moving $\vec{v}\cdot\hat{n} = U_n$

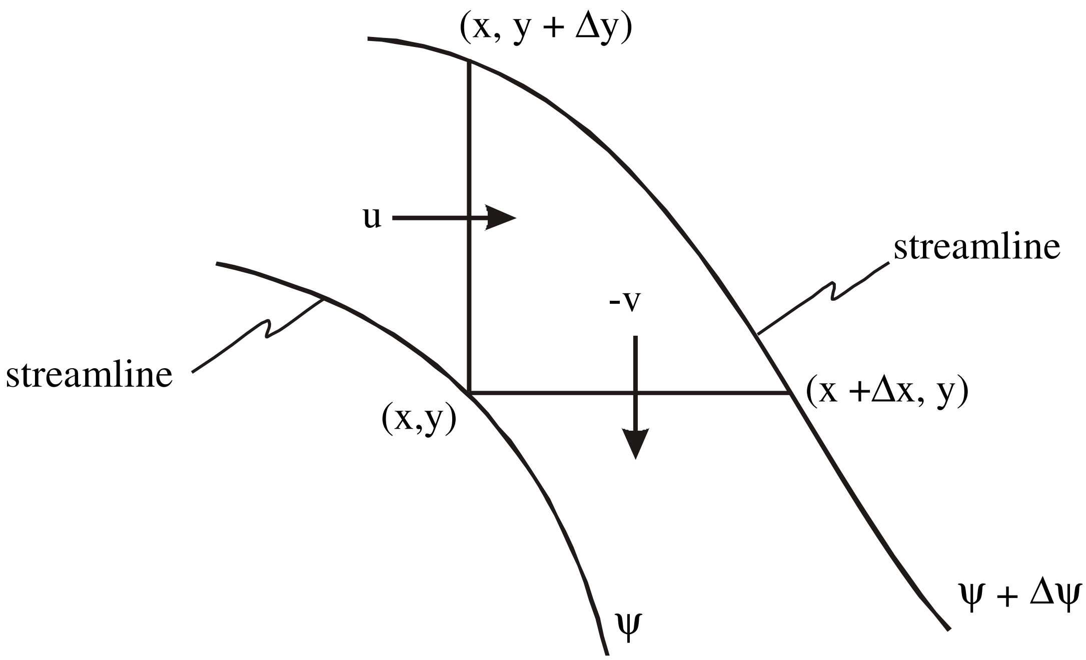
$$\psi = \psi_0 + \int \underbrace{U_n}_{given} d\ell \quad \text{on the boddy}$$





Flux $\Delta\psi = -v\Delta x = u\Delta y$.

Therefore, $u = \dfrac{\partial\psi}{\partial y}$ and $v = -\dfrac{\partial\psi}{\partial x}$



Summary of velocity potential formulation vs. stream-function formulation for ideal flows

$$\left\{ \begin{array}{lll} \text{For irrotational flow} & \text{use} & \phi \\ \text{For incompressible flow} & \text{use} & \psi \\ \text{For P-Flow} & \text{use} & \phi \text{ or } \psi \end{array} \right\}$$

	velocity potential	stream-function
definition	$\vec{v} = \nabla\phi$	$\vec{v} = \nabla \times \vec{\psi}$
continuity $\nabla \cdot \vec{v} = 0$	$\nabla^2\phi = 0$	automatically satisfied
irrotationality $\nabla \times \vec{v} = 0$	automatically satisfied	$\nabla \times \left(\nabla \times \vec{\psi}\right) = \nabla\left(\nabla \cdot \vec{\psi}\right) - \nabla^2\vec{\psi} = 0$
2D: $w = 0, \frac{\partial}{\partial z} = 0$		
continuity	$\nabla^2\phi = 0$	automatically satisfied
irrotationality	automatically satisfied	$\psi \equiv \psi_z : \nabla^2\psi = 0$

Cauchy-Riemann equations for $(\phi,\ \psi)$ = (real, imaginary) part of an analytic complex function of $z = x + iy$

Cartesian (x, y)	$u = \frac{\partial\phi}{\partial x}$ $v = \frac{\partial\phi}{\partial y}$	$u = \frac{\partial\psi}{\partial y}$ $v = -\frac{\partial\psi}{\partial x}$
Polar (r,θ)	$v_r = \frac{\partial\phi}{\partial r}$ $v_\theta = \frac{1}{r}\frac{\partial\phi}{\partial\theta}$	$v_r = \frac{1}{r}\frac{\partial\psi}{\partial\theta}$ $v_\theta = -\frac{\partial\psi}{\partial r}$

Given ϕ or ψ for 2D flow, use Cauchy-Riemann equations to find the other:

e.g. If $\phi = xy$, then $\psi = ?$

$$\left. \begin{array}{lll} u = \dfrac{\partial\phi}{\partial x} = y = \dfrac{\partial\psi}{\partial y} & \rightarrow & \psi = \dfrac{1}{2}y^2 + f_1(x) \\ v = \dfrac{\partial\phi}{\partial y} = x = -\dfrac{\partial\psi}{\partial x} & \rightarrow & \psi = -\dfrac{1}{2}x^2 + f_2(y) \end{array} \right\} \Rightarrow \psi = \frac{1}{2}(y^2 - x^2) + const$$