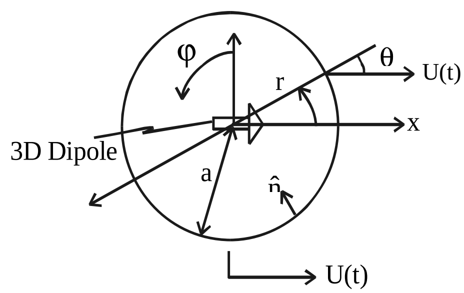
2.20 - Marine Hydrodynamics
Lecture 13

3.18 Unsteady Motion - Added Mass

D'Alembert: ideal, irrotational, unbounded, steady.

Example Force on a sphere accelerating ($U = U(t)$, unsteady) in an unbounded fluid that is at at rest at infinity.



K.B.C on sphere: $\left.\frac{\partial \phi}{\partial r}\right|_{r=a} = U(t)\cos\theta$

Solution: Simply a 3D dipole (no stream)

$$\phi = -U(t)\frac{a^3}{2r^2}\cos\theta$$

Check: $\left.\frac{\partial \phi}{\partial r}\right|_{r=a} = U(t)\cos\theta$

Hydrodynamic force:

$$F_x = -\rho \iint_B \left(\frac{\partial \phi}{\partial t} + \frac{1}{2} \left| \nabla \phi \right|^2 \right) n_x dS$$

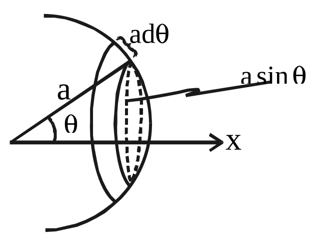
On r = a,

$$\left. \frac{\partial \phi}{\partial t} \right|_{r=a} = -\dot{U} \frac{a^3}{2r^2} \cos \theta|_{r=a} = -\frac{1}{2} \dot{U} a \cos \theta$$

$$\nabla \phi|_{r=a} = \left(\frac{\partial \phi}{\partial r}, \frac{1}{r} \frac{\partial \phi}{\partial \theta}, \frac{1}{r \sin \theta} \frac{\partial \phi}{\partial \varphi} \right) = \left(U \cos \theta, \frac{1}{2} U \sin \theta, 0 \right)$$

$$\left. \left| \nabla \phi \right|^2 \right|_{r=a} = U^2 \cos^2 \theta + \frac{1}{4} U^2 \sin^2 \theta ; \hat{n} = -\hat{e}_r, n_x = -\cos \theta$$

$$\iint_B dS = \int_0^\pi (a d\theta) (2\pi a \sin \theta)$$



Finally,

$$F_x = (-\rho)\, 2\pi a^2 \int_0^\pi d\theta\, (\sin\theta) \left(\underbrace{-\cos\theta}_{n_x} \right) \left[\underbrace{-\frac{1}{2}\, \dot{U} a \cos\theta}_{\frac{\partial\phi}{\partial t}} + \frac{1}{2} \left(\underbrace{U^2 \cos^2\theta + \frac{1}{4} U^2 \sin^2\theta}_{|\nabla\phi|^2} \right) \right]$$

$$F_x = -\dot{U}(\rho a^3)\pi \underbrace{\int_0^\pi d\theta \sin\theta \cos^2\theta}_{2/3} \quad + \quad (\rho U^2)\pi a^2 \underbrace{\int_0^\pi d\theta \sin\theta \cos\theta \left(\cos^2\theta + \frac{1}{4}\sin^2\theta \right)}_{= 0, \text{ D'alembert revisited}}$$

$$\underbrace{F_x}_{\text{Hydrodynamic Force}} = - \underbrace{\dot{U}(t)}_{\text{Acceleration}} \left[\underbrace{\rho}_{\textbf{Fluid}\text{ Density}} \quad \underbrace{\frac{2}{3}\pi a^3}_{\text{Volume } = 1/2\forall_{sphere}} \right]$$

Thus the **Hydrodynamic Force** on a sphere of diameter a moving with velocity $U(t)$ in an unbounded fluid of density ρ is given by

$$\boxed{F_x = -\dot{U}(t) \left[\rho \frac{2}{3} \pi a^3 \right]}$$

Comments:

- If $\dot{U} = 0 \rightarrow F_x = 0$, i.e., steady translation $\rightarrow$ no force (D'Alembert's Condition ok).
- $F_x \propto \dot{U}$ with a $(-)$ sign, i.e., the fluid tends to 'resist' the acceleration.
- $[\cdots]$ has the units of (**fluid**) mass $\equiv m_a$
- Equation of Motion for a body of mass M that moves with velocity U:

  $$\underbrace{M}_{\text{Body mass}} \dot{U} = \Sigma F = \underbrace{F_H}_{\substack{\text{Hydrodynamic force} \\ \iint_S p\hat{n} dS}} + \underbrace{F_B}_{\text{All other forces on body}} = \big(-\dot{U} \underbrace{m_a}_{\textbf{Fluid}\text{ mass}} \big) + F_B \Leftrightarrow$$

  $$\boxed{(M + m_a)\, \dot{U} = F_B}$$

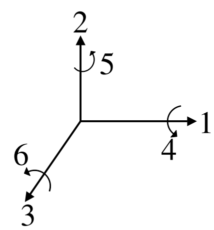
  i.e., the presence of fluid around the body acts as an **added** or **virtual** mass to the body.

3.19 General 6 Degrees of Freedom Motions

3.19.1 Notation Review

(3D) U_1, U_2, U_3: Translational velocities

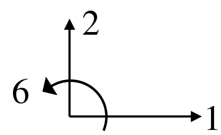
$U_4 \equiv \Omega_1$, $U_5 \equiv \Omega_2$, $U_6 \equiv \Omega_3$: Rotational velocities



(2D) U_1, U_2: Translational velocities

$U_6 \equiv \Omega_3$: Rotational velocity

$U_3 = U_4 = U_5 = 0$



3.19.2 Added Mass Tensor (matrix)

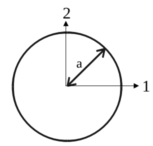
$$m_{ij} \ ; i, j = 1, 2, 3, 4, 5, 6$$

m_{ij}: associated with force on body in i direction due to unit acceleration in j direction. For example, for a sphere:

$$m_{11} = m_{22} = m_{33} = {}^1\!/_2 \rho \forall = (m_A) \quad \text{all other } m_{ij} = 0$$

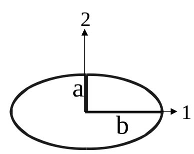
3.19.3 Added Masses of Simple 2D Geometries

- Circle



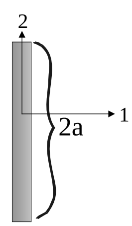
$$m_{11} = m_{22} = \rho\forall = \rho\pi a^2$$

- Ellipse



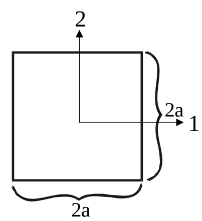
$$m_{11} = \rho\pi a^2, m_{22} = \rho\pi b^2$$

- Plate



$$m_{11} = \rho\pi a^2, m_{22} = 0$$

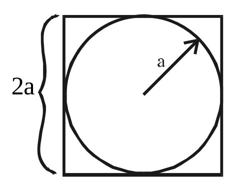
- Square



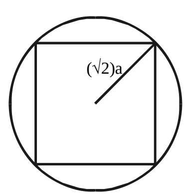
$$m_{11} = m_{22} \approx 4.754\rho a^2$$

A reasonable approximation to estimate the added mass of a 2D body is to use the displaced mass ($\rho\forall$) of an 'equivalent cylinder' of the same lateral dimension or one that 'rounds off' the body. For example, consider a square and approximate with an

(a) inscribed circle: $m_A = \rho\pi a^2 = 3.14\rho a^2$.



(b) circumscribed circle: $m_A = \rho\pi\left(\sqrt{2}a\right)^2 = 6.28\rho a^2$.

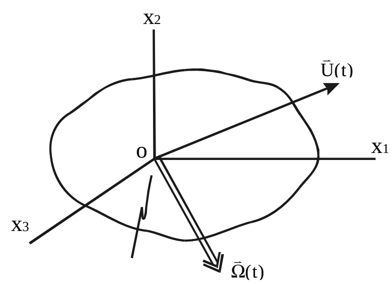


Arithmetic mean of (a) + (b) $\approx 4.71\rho a^2$.

3.19.4 **Generalized Forces and Moments**

In this paragraph we are looking at the most general case where forces and moments are induced on rigid body moving with 6 DoF motions, in an unbounded fluid that is at rest at infinity.

Body fixed reference frame, i.e., $OX_1X_2X_3$ is **fixed** on the body.



$$\vec{U}(t) = (U_1, U_2, U_3)\,, \quad \text{translational velocity}$$
$$\vec{\Omega}(t) = (\Omega_1, \Omega_2, \Omega_3) \equiv (U_4, U_5, U_6)\,, \quad \text{rotational velocity with respect to O}$$

Consider a body with a 6 DoF motion $(\vec{U}, \vec{\Omega})$, and a fixed reference frame $OX_1X_2X_3$. Then the hydrodynamic forces and moments with respect to O are given by the following relations (JNN §4.13)

- Forces

$$F_j = -\underset{1.}{\dot{U}_i m_{ji}} - E_{jkl}\, \underset{2.}{U_i \Omega_k}\, m_{li} \quad \text{with} \quad i = 1,2,3,4,5,6 \quad \text{and} \quad j,k,l = 1,2,3$$

- Moments

$$M_j = -\underset{3.}{\dot{U}_i m_{j+3,i}} - E_{jkl}\, \underset{2.}{U_i \Omega_k}\, m_{l+3,i} - E_{jkl}\, \underset{3.}{U_k U_i}\, m_{li} \quad \text{with} \quad i = 1,2,3,4,5,6 \quad \text{and} \quad j,k,l = 1,2,3$$

Einstein's Σ notation applies.

$$E_{jkl} = \text{`alternating tensor'} = \begin{cases} 0 & \text{if any } j,k,l \text{ are equal} \\ 1 & \text{if} j,k,l \text{ are in cyclic order, i.e.,} \\ & \quad (1,2,3),(2,3,1), \text{or } (3,1,2) \\ -1 & \text{if} j,k,l \text{ are not in cyclic order i.e.,} \\ & \quad (1,3,2),(2,1,3),(3,2,1) \end{cases}$$

Note:

(a) if $\Omega_k \equiv 0$, $F_j = -\dot{U}_i m_{ji}$ (as expected by definition of m_{ij}).
Also if $\dot{U}_i \equiv 0$, then $F_j = 0$ for any U_i, no force in steady translation.

(b) $B_l \sim U_i m_{li}$ 'added momentum' due to rotation of axes.

Then all the terms marked as 2. are proportional to $\sim \vec{\Omega} \times \vec{B}$ where $\vec{B}$ is linear momentum (momentum from i coordinate into new x_j direction).

(c) If $\Omega_k \equiv 0 : M_j = -\dot{U}_i m_{j+3,i} m_{ij} - \underbrace{E_{jkl} U_k U_i m_{li}}_{\text{even with } \dot{U}=0,\ M_j \neq 0 \text{ due to this term}}$.

Moment on a body due to pure steady translation – 'Munk' moment.

3.19.5 **Example** Generalized motions, forces and moments.

A certain body has non-zero added mass coefficients only on the diagonal, i.e. $m_{ij} = \delta_{ij}$. For a body motion given by $U_1 = t$, $U_2 = -t$, and all other U_i, $\Omega_i = 0$, the forces and moments on the body in terms of m_i are:

$F_1 =$ _____, $F_2 =$ _____, $F_3 =$ _____, $M_1 =$ _____, $M_2 =$ _____, $M_3 =$ _____

Solution:

$m_{ij} = \delta_{ij}$

$U_1 = t \quad U_2 = -t \quad U_i = 0 \quad i = 3, 4, 5, 6 \quad \Omega_k = 0 \quad k = 1, 2, 3$

$\dot{U}_1 = 1 \quad \dot{U}_2 = -1 \quad \dot{U}_i = 0 \quad i = 3, 4, 5, 6$

Use the relations from (JNN §4.13):

$$
\begin{aligned}
F_j &= -\dot{U}_i m_{ij} - E_{jkl} U_i \Omega_k m_{il} \xrightarrow{\Omega_k=0} \\
F_j &= -\dot{U}_i m_{ij}
\end{aligned}
$$

$$
\begin{aligned}
M_j &= -\dot{U}_i m_{i(j+3)} - E_{jkl} U_i \Omega_k m_{i(l+3)} - E_{jkl} U_k U_i m_{li} \xrightarrow{\Omega_k=0} \\
M_j &= -\dot{U}_i m_{i(j+3)} - E_{jkl} U_k U_i m_{li}
\end{aligned}
$$

where $i = 1, 2, 3, 4, 5, 6$ and $j, k, l = 1, 2, 3$

For F_1, F_2, F_3 use the previous relationship for F_j with $j = 1, 2, 3$ respectively:

$$
F_1 = -\underbrace{\dot{U}_1}_{=1} m_{11} - \dot{U}_2 \underbrace{m_{21}}_{=0} - \dot{U}_3 \underbrace{m_{31}}_{=0} - \dot{U}_4 \underbrace{m_{41}}_{=0} - \dot{U}_5 \underbrace{m_{51}}_{=0} - \dot{U}_6 \underbrace{m_{61}}_{=0} \rightarrow \boxed{F_1 = -m_{11}}
$$

$$
F_2 \overset{\text{Check}}{=} -\underbrace{\dot{U}_2}_{=-1} m_{22} \rightarrow \boxed{F_2 = m_{22}}
$$

$$
F_3 \overset{\text{Check}}{=} -\underbrace{\dot{U}_3}_{=0} m_{33} \rightarrow \boxed{F_3 = 0}
$$

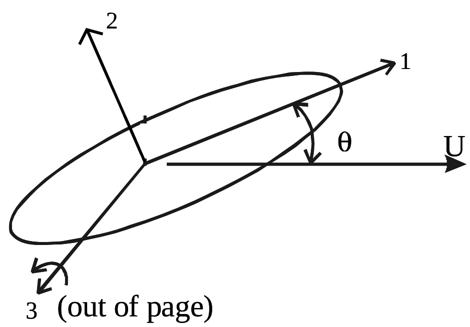
For $M1, M_2, M_3$ use the previous relationship for M_j with $j = 1, 2, 3$ respectively:

$$
\begin{aligned}
M_1 &= -\dot{U}_i m_{i(1+3)} - E_{1kl} U_k U_i m_{li} \\
&= -\dot{U}_i m_{i4} - E_{1kl} U_k U_i m_{li} \\
&= -\dot{U}_1 \underbrace{m_{14}}_{=0} - \dot{U}_2 \underbrace{m_{24}}_{=0} - \dot{U}_3 \underbrace{m_{34}}_{=0} - \underbrace{\dot{U}_4}_{=0} m_{44} - \dot{U}_5 \underbrace{m_{54}}_{=0} - \dot{U}_6 \underbrace{m_{64}}_{=0} \\
&\quad - E_{123} U_2 \big(U_1 \underbrace{m_{13}}_{=0} + U_2 \underbrace{m_{23}}_{=0} + \underbrace{U_3}_{=0} m_{33} + U_4 \underbrace{m_{43}}_{=0} + U_5 \underbrace{m_{53}}_{=0} + U_6 \underbrace{m_{63}}_{=0} \big) \\
&\quad - E_{132} \underbrace{U_3}_{=0} \big(U_1 \underbrace{m_{12}}_{=0} + \underbrace{U_2}_{=-1} m_{22} + U_3 \underbrace{m_{32}}_{=0} + U_4 \underbrace{m_{42}}_{=0} + U_5 \underbrace{m_{52}}_{=0} + U_6 \underbrace{m_{62}}_{=0} \big) \rightarrow \boxed{M_1 = 0}
\end{aligned}
$$

$$
\begin{aligned}
M_2 &= -\dot{U}_i m_{i5} - E_{2kl} U_k U_i m_{li} \\
&= \dot{U}_5 m_{55} - E_{231} U_3 U_i m_{1i} - E_{213} U_1 U_i m_{3i} \\
&= -E_{213} U_1 U_3 m_{33} \rightarrow \boxed{M_2 = 0}
\end{aligned}
$$

$$
\begin{aligned}
M_3 &= -\dot{U}_i m_{i6} - E_{3kl} U_k U_i m_{li} \\
&= \dot{U}_6 m_{66} - \underbrace{E_{312}}_{+1} U_1 U_i m_{2i} - \underbrace{E_{321}}_{-1} U_2 U_i m_{1i} \\
&= - \underbrace{U_1}_{t} \underbrace{U_2}_{-t} m_{22} + \underbrace{U_2}_{-t} \underbrace{U_1}_{t} m_{11} \rightarrow \boxed{M_3 = t^2 (m_{22} - m_{11})}
\end{aligned}
$$

3.19.6 **Example** Munk Moment on a 2D submarine in steady translation



$$U_1 = U\cos\theta$$
$$U_2 = -U\sin\theta$$

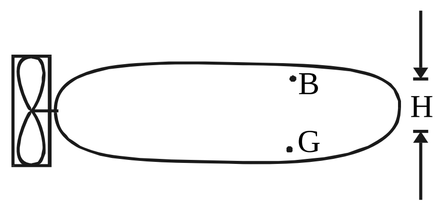
Consider steady translation motion: $\dot{U} = 0; \Omega_k = 0$. Then

$$M_3 = -E_{3kl}U_kU_im_{li}$$

For a 2D body, $m_{3i} = m_{i3} = 0$, also $U_3 = 0, i, k, l = 1, 2$. This implies that:

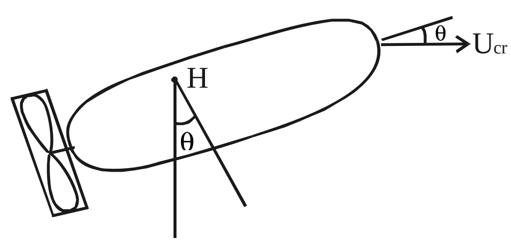
$$\begin{aligned} M_3 &= -\underbrace{E_{312}}_{=1}U_1\left(U_1m_{21} + U_2m_{22}\right) - \underbrace{E_{321}}_{=-1}U_2\left(U_1m_{11} + U_2m_{12}\right) \\ &= -U_1U_2\left(m_{22} - m_{11}\right) \\ &= U^2\sin\theta\cos\theta\left(\underbrace{m_{22} - m_{11}}_{>0}\right) \end{aligned}$$

Therefore, $M_3 > 0$ for $0 < \theta < \pi/2$ ('Bow up'). Therefore, a submarine under forward motion is unstable in pitch (yaw). For example, a small bow-up tends to grow with time, and control surfaces are needed as shown in the following figure.



- Restoring moment $\approx (\rho \mathrm{g} \forall H)\sin\theta$.
- critical speed U_{cr} given by:

$$(\rho g \forall)\, H \sin\theta \geq U_{cr}^2 \sin\theta \cos\theta \,(m_{22} - m_{11})$$



Usually $m_{22} >> m_{11}, m_{22} \approx \rho\forall$. For small $\theta, \cos\theta \approx 1$. So, $U_{cr}^2 \leq gH$ or $F_{cr} \equiv \frac{U_{cr}}{\sqrt{gH}} \leq 1$. Otherwise, control fins are required.