2.161 Signal Processing: Continuous and Discrete
Fall 2008

MASSACHUSETTS INSTITUTE OF TECHNOLOGY
DEPARTMENT OF MECHANICAL ENGINEERING

2.161 *Signal Processing - Continuous and Discrete*
Fall Term 2008

Lecture 1[1]

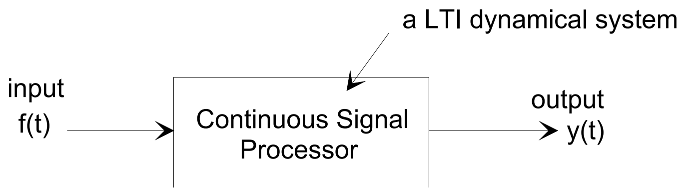
Reading:

- Class handout: *The Dirac Delta and Unit-Step Functions*

1 Introduction to Signal Processing

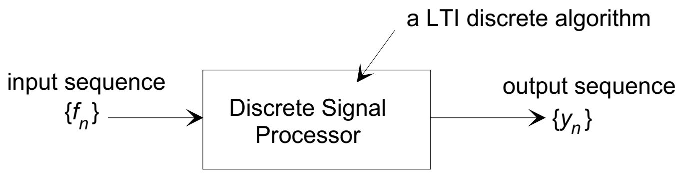
In this class we will primarily deal with processing time-based functions, but the methods will also be applicable to spatial functions, for example image processing. We will deal with

(a) Signal processing of continuous waveforms $f(t)$, using continuous LTI systems (filters).



and

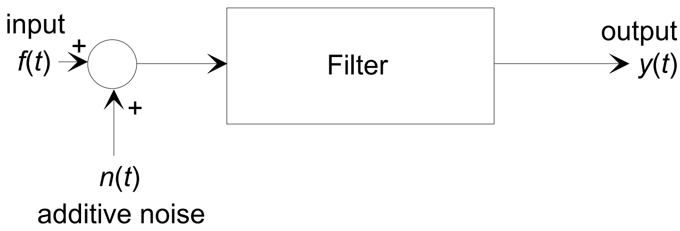
(b) Discrete-time (digital) signal processing of data sequences $\{f_n\}$ that might be samples of real continuous experimental data, such as recorded through an analog-digital converter (ADC), or implicitly discrete in nature.



Some typical applications that we look at will include

(a) Data analysis, for example estimation of spectral characteristics, delay estimation in echolocation systems, extraction of signal statistics.

(b) Signal enhancement. Suppose a waveform has been contaminated by additive "noise", for example 60Hz interference from the ac supply in the laboratory.

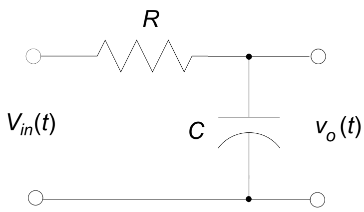


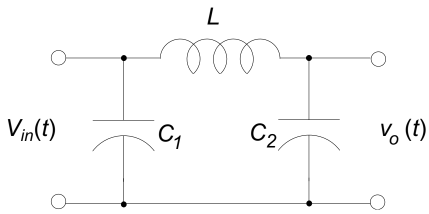
The task is to design a filter that will minimize the effect of the interference while not destroying information from the experiment.

(c) Signal detection. Given a noisy experimental record, ask the question whether a known signal is present in the data.

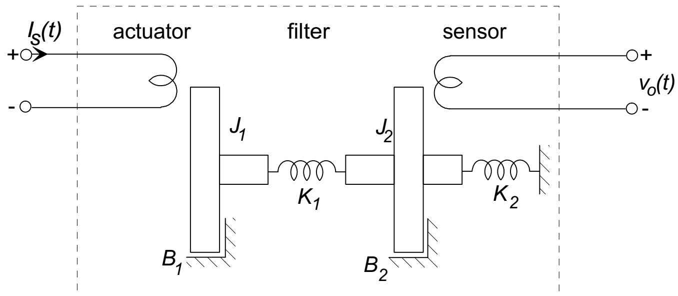
1.1 Processing methods

(a) Passive Continuous Filters: We will investigate signal processing using passive continuous LTI (Linear Time-Invariant) dynamical systems. These will typically be electrical R-L-C systems, for example

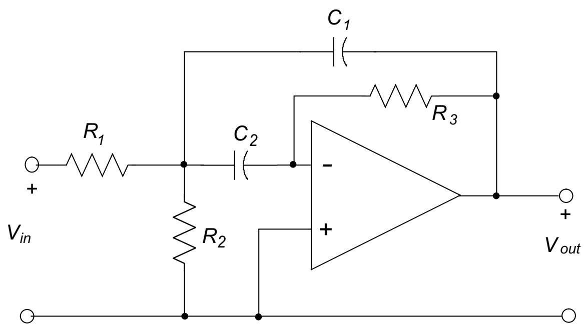




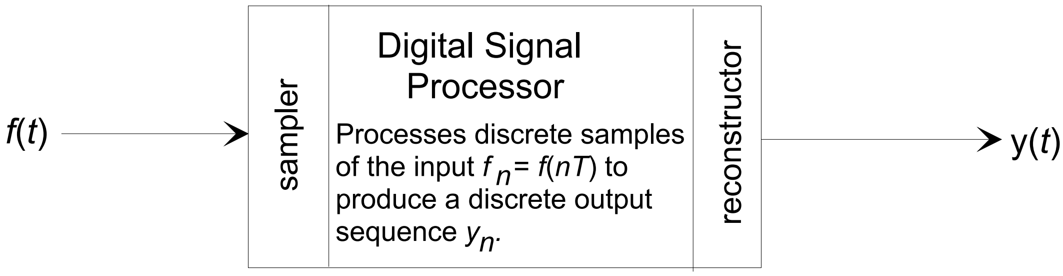
or even an electro mechanical system using rotational elements:



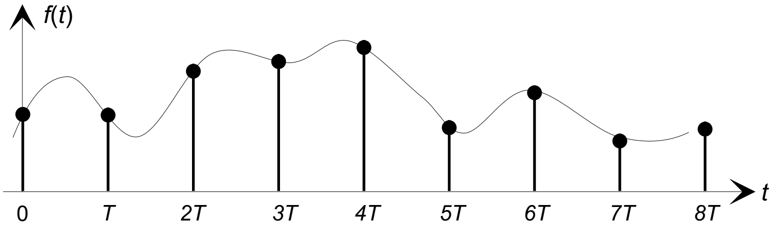
(b) Active Continuous Filters: Modern continuous filters are implemented using operational amplifiers. We will investigate simple op-amp designs.



(c) Digital Signal Processors: Here a digital system (a computer or DSP chip) is used to process a data stream.



(i) The sampler (A/D converter) records the signal value at discrete times to produce a sequence of samples $\{f_n\}$ where $f_n = f(nT)$ (T is the sampling interval.



(ii) At each interval, the output sample y_n is computed, based on the history of the input and output, for example

$$y_n = \frac{1}{3}\left(f_n + f_{n-1} + f_{n-2}\right)$$

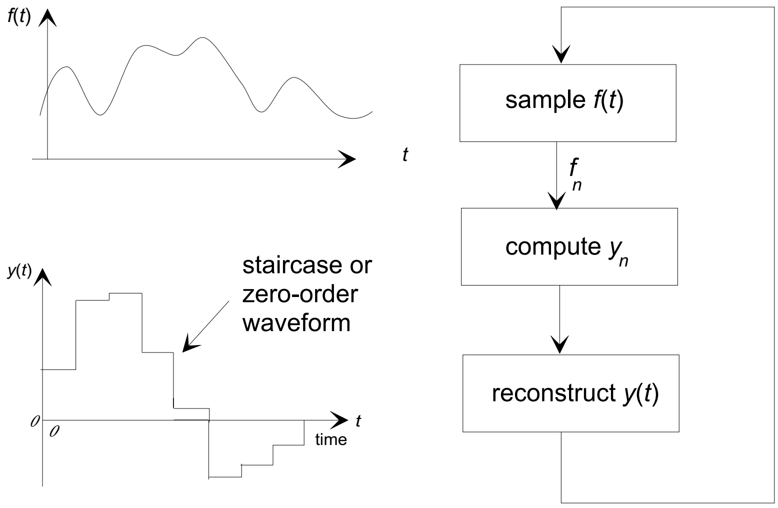
3-point moving average filter, and

$$y_n = 0.8y_{n-1} + 0.2f_n$$

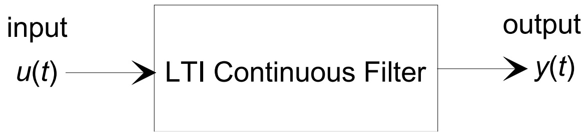
is a simple recursive first-order low-pass digital filter. Notice that they are *algorithms.*

(iii) The reconstructor takes each output sample and creates a continuous waveform.

In real-time signal processing the system operates in an infinite loop:



2 Properties of LTI Continuous Filters



A LTI filter is dynamical SISO (single-input single-output) linear system, governed by an ODE with constant coefficients. From elementary linear system theory, some fundamental properties of LTI systems are:

(a) The Principle of Superposition This is the fundamental property of linear systems. For a system at rest at time $t = 0$, if the response to input $f_1(t)$ is $y_1(t)$, and the response to $f_2(t)$ is $y_2(t)$, then the response to a linear combination of $f_1(t)$ and $f_2(t)$, that is $f_3(t) = af_1(t) + bf_2(t)$ (a and b constants) is

$$y_3(t) = ay_1(t) + by_2(t).$$

(b) The Differentiation Property If the response to input $f(t)$ is $y(t)$, then the response to the derivative of $f(t)$, that is $f_1(t) = df/dt$ is

$$y_1(t) = \frac{dy}{dt}.$$

(c) The Integration Property If the response to input $f(t)$ is $y(t)$, then the response to the integral of $f(t)$, that is $f_1(t) = \int_{-\infty}^{t} f(t)dt$ is

$$y_1(t) = \int_{-\infty}^{t} y(t)dt.$$

(d) Causality A *causal* system is non-anticipatory, that is it does not respond to an input before it occurs. Physical LTI systems are causal.

3 The Dirac Delta Function

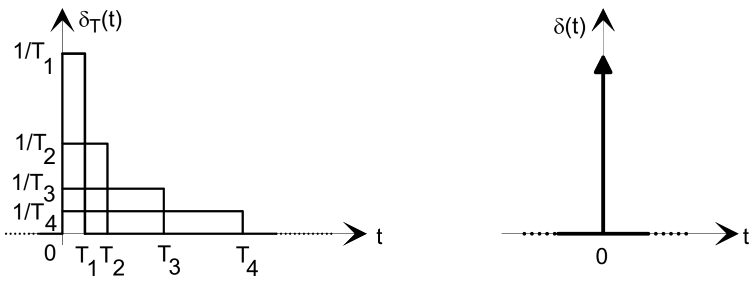
The Dirac delta function is a non-physical, singularity function with the following definition

$$\delta(t) = \begin{cases} 0 & \text{for } t \neq 0 \\ \text{undefined} & \text{at } t = 0 \end{cases}$$

but with the requirement that

$$\int_{-\infty}^{\infty} \delta(t)dt = 1,$$

that is, the function has unit area. Despite its name, the delta function is not truly a function. Rigorous treatment of the Dirac delta requires *measure theory* or the *theory of distributions.*



a) Unit pulses of different extents   b) The impulse function

The figure below shows a *unit pulse* function $\delta_T(t)$, that is a brief rectangular pulse function of extent T, defined to have a constant amplitude $1/T$ over its extent, so that the area $T \times 1/T$ under the pulse is unity:

$$\delta_T(t) = \begin{cases} 0 & \text{for } t \le 0 \\ 1/T & 0 < t \le T \\ 0 & \text{for } t > T. \end{cases}$$

The Dirac delta function (also known as the impulse function) can be defined as the limiting form of the unit pulse $\delta_T(t)$ as the duration T approaches zero. As the extent T of $\delta_T(t)$ decreases, the amplitude of the pulse increases to maintain the requirement of unit area under the function, and

$$\delta(t) = \lim_{T \to 0} \delta_T(t).$$

The impulse is therefore defined to exist only at time $t = 0$, and although its value is strictly undefined at that time, it must tend toward infinity so as to maintain the property of unit area in the limit.

4 Properties of the Delta Function

4.0.1 Time Shift

An impulse occurring at time $t = a$ is $\delta(t - a)$.

4.0.2 The strength of an impulse

Because the amplitude of an impulse is infinite, it does not make sense to describe a scaled impulse by its amplitude. Instead, the *strength* of a scaled impulse $K\delta(t)$ is defined by its area K.

4.0.3 The "Sifting" Property of the Impulse

When an impulse appears in a product within an integrand, it has the property of "sifting" out the value of the integrand at the time of its occurrence:

$$\int_{-\infty}^{\infty} f(t)\delta(t - a)dt = f(a)$$

This is easily seen by noting that $\delta(t-a)$ is zero except at $t=a$, and for its infinitesimal duration $f(t)$ may be considered a constant and taken outside the integral, so that

$$\int_{-\infty}^{\infty} f(t)\delta(t-a)dt = f(a)\int_{-\infty}^{\infty} \delta(t-a)dt = f(a)$$

from the unit area property.

4.0.4 Scaling

A helpful identity is the scaling property:

$$\int_{-\infty}^{\infty} \delta(\alpha t)dt = \int_{-\infty}^{\infty} \delta(u)\frac{du}{|\alpha|} = \frac{1}{|\alpha|}$$

and so

$$\delta\left(\alpha t\right) = \frac{1}{|\alpha|}\delta(t).$$

4.0.5 Laplace Transform

$$\mathcal{L}\left\{\delta(t)\right\} = \int_{0^-}^{\infty} \delta(t)e^{-st}dt = 1$$

by the sifting property.

4.0.6 Fourier Transform

$$\mathcal{F}\left\{\delta(t)\right\} = \int_{-\infty}^{\infty} \delta(t)e^{-j\Omega t}dt = 1$$

by the sifting property.

5 Practical Applications of the Dirac Delta Function

- The most important application of δt in linear system theory is directly related to its Laplace transform property, $\mathcal{L}\left\{\delta(t)\right\} = 1$. Consider a SISO LTI system with transfer function $H(s)$, with input $u(t)$ and output $y(t)$, so that in the Laplace domain

  $$Y(s) = H(s)U(s).$$

  If the input is $u(t) = \delta(t)$, so that $U(s) = 1$, then $Y(s) = H(s).1$, and through the inverse Laplace transform

  $$y(t) = h(t) = \mathcal{L}^{-1}\left\{H(s)\right\}.$$

  where $h(t)$ is defined as the system's *impulse response*. The impulse response completely characterizes the system, in the sense that it allows computation of the transfer function (and hence the differential equation).

- The impulse response $h(t)$ is used in the convolution integral.
- In signal processing the delta function is used to create a Dirac comb (also known as an impulse train, or Shah function):

  $$\Delta_T(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT)$$

  is used in sampling theory. A continuous waveform $f(t)$ is *sampled* by multiplication by the Dirac comb

  $$f^*(t) = f(t)\Delta_T(t) = \sum_{n=-\infty}^{\infty} f(t - nT)\delta(t - nT),$$

  where $f^*(t)$ is the sampled waveform, producing a train of weighted impulses.