2.161 Signal Processing: Continuous and Discrete
Fall 2008

MASSACHUSETTS INSTITUTE OF TECHNOLOGY

DEPARTMENT OF MECHANICAL ENGINEERING

*2.161 Signal Processing - Continuous and Discrete*
Fall Term, 2007

# Quiz #1 — October 18, 2007

<u>Notes:</u>

- The quiz is open-book.
- The time allowed is ninety minutes.
- There are six problems, answer them all.
- Partial credit will be given.

Problem 1: (15 points)

Show that to interpolate a value to the mid-point of a sampling interval in the sample set $\{f_n\}$, $n = 0 \dots N-1$, where $f_n = f(n\Delta T)$, the cardinal (Whittaker) reconstructor may be written

$$f(n\Delta T + \Delta T/2) = \frac{1}{\pi} \sum_{k=0}^{N-1} f_k \frac{(-1)^k}{(n-k+1/2)}$$

Problem 2: (20 points)

**(a)** Design (find the transfer function of) a unity-gain 3rd-order low-pass Butterworth analog filter with a -3dB cut-off frequency of 10 rad/s.

**(b)** Make a pole-zero plot for your filter.

**(c)** Convert your design to a high-pass filter with the same cut-off frequency.

**(d)** If (do not do it) you were to convert your design to a band-pass filter, what would be the order of the filter, how many zeros would be created, and where in the $s$-plane would the zeros lie?

Problem 3: (15 points)

Many real-life signal processing problems involve waveforms containing echoes, or reverberation. Consider a continuous-time linear filter with impulse response

$$h(t) = \delta(t) + \delta(t - \tau)$$

Assume that the filter is excited with waveform $f(t)$, and that the output is $y(t)$

**(i)** Show that the output contains an echo, that is

$$y(t) = f(t) + f(t - \tau)$$

**(ii)** Find the frequency response $H(j\omega)$ for the filter, and express it in terms of its magnitude $|H(j\omega)|$, and phase $\angle H(j\omega)$.
**Hint:**

$$1 + e^{-j\theta} = e^{-j\theta/2}\left[e^{j\theta/2} + e^{-j\theta/2}\right]$$

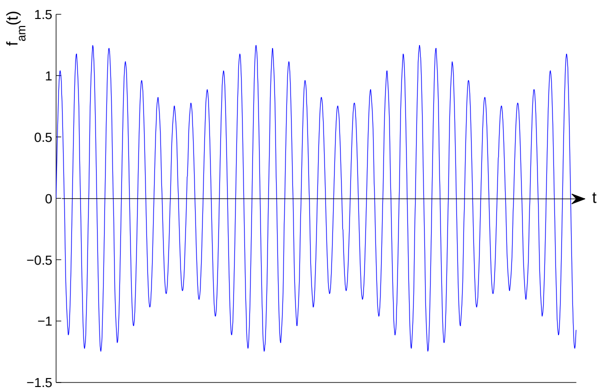
**(iii)** Assume $f(t) = \sin \omega t$. At what values of $\omega$ will the filter exhibit no steady-state output?

Problem 4: (20 points)

An AM (amplitude-modulated) radio signal $f_{AM}(t)$ is described by

$$f_{AM}(t) = (1 + a f_{audio}(t)) \sin(\omega_c t)$$

where $f_{audio}(t)$ is the audio signal, $\sin(\omega_c t)$ is known as radio-frequency *carrier* signal ($f_c =$ 500 – 1600 kHz - the AM band), and $a$ is a positive constant that determines the *modulation depth*. (Note that we require $|a f_{audio}(t)| < 1$ otherwise we have over-modulation.) the following figure shows an AM signal with an "audio" waveform that is a simple low frequency sinusoid. You can see how the audio signal "modulates" the amplitude of the rf signal.



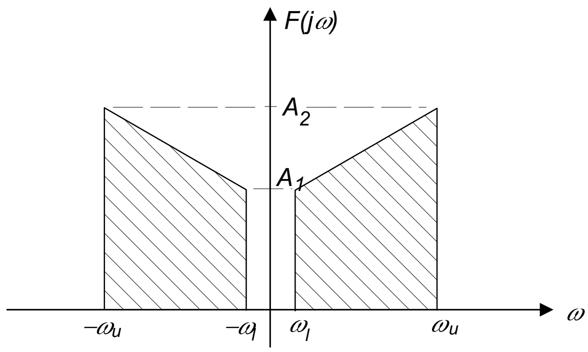

**(a)** Sketch the magnitude of the Fourier transform of $f_{AM}(t)$ when $f_{audio}(t) = 0$.

**(b)** Let $a = 0.5$, and sketch the magnitude of the Fourier transform of $f_{AM}(t)$ when

$$f_{audio}(t) = 0.5\cos(2\pi \cdot 1000t) + 0.25\cos(2\pi \cdot 2000t)$$

(**Hint:** There is no need to actually compute the FT. Consider expanding $f_{AM}(t)$, or simply use properties of the FT.)

**(c)** Use your result from (b) to generalize, and sketch the magnitude spectrum of $f_{AM}(t)$ when $f_{audio}(t)$ has a spectrum (again let $a = 0.5$):



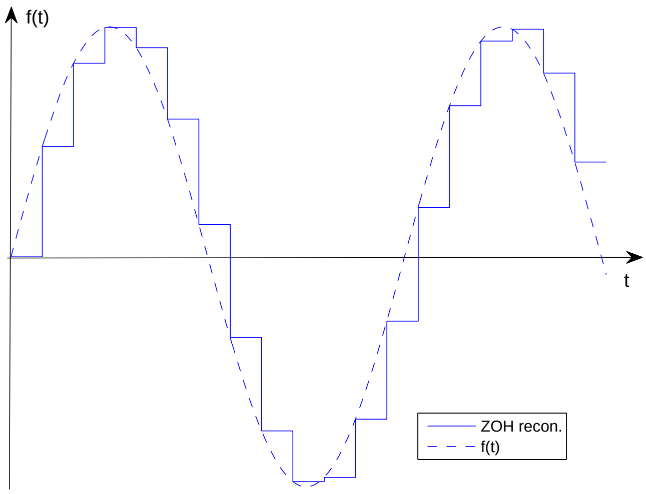

**(d)** If $f_{audio}$ has a bandwidth $B = \omega_u - \omega_l$, what is the bandwidth of a band-pass filter that would be necessary to select the signal $f_{AM}(t)$ out of all the other AM radio stations?

Problem 5: (15 points)

The most commonly used (approximate) data reconstructor is the "zero-order hold" (ZOH) which simply "holds" the output value $y(n\Delta T)$ as a constant over the reconstruction interval:

$$y(t) = f(n\Delta T) \qquad n\Delta T \leq t < (n+1)\Delta T$$

where $f(n\Delta T)$ is the value of input at time $t = n\Delta T$. The output of the ZOH therefore looks like a staircase function, with discrete jumps to a new value at each update time $n\Delta T$:




**(a)** Show that the ZOH can be represented as a linear filter with a pulse-like impulse response

$$\begin{aligned} h(t) &= 1 \qquad 0 \leq t < \Delta T \\ &= 0 \qquad \text{otherwise} \end{aligned}$$

Hint: Consider the sampled input data waveform $f^*(t)$ as a weighted impulse train at intervals $\Delta T$,

$$f^*(t) = \sum_{n=-\infty}^{\infty} f(nT)\delta(t - n\Delta T).$$

**(b)** Find and sketch the frequency response function $H(j\omega)$ for the ZOH data reconstructor.

**(c)** Compare $H(j\omega)$ with the frequency response of the ideal (cardinal) data reconstructor, and comment on why the ZOH is a non-ideal reconstructor.

**(d)** The ZOH reconstruction (see the fig above) seems to be slightly delayed from $f(t)$. Determine the delay.

Problem 6: (15 points)

The significant frequency range of an analog signal extends to 10 kHz. Beyond 10 kHz the signal spectrum rolls-off (attenuates) at a rate of 20 dB/decade.

The signal is to be sampled at a rate of 200 kHz. The aliased frequency components introduced into the 10 kHz range of interest must be kept below -60 dB as compared to signal components.

Suppose we use an analog low-pass pre-aliasing filter whose passband is flat over the 10 kHz band, and then attenuates at a rate steep enough to satisfy the above sampling requirements. What is this attenuation rate in dB/decade? What would be the minimum order of a low-pass filter to satisfy this condition?

**Some useful/useless information:**

| $\theta$ | $\sin\theta$ | $\cos\theta$ | $\tan\theta$ |
|---|---|---|---|
| 0 | 0 | 1 | 0 |
| $\pi/6$ | $1/2$ | $\sqrt{3}/2$ | $\sqrt{3}/3$ |
| $\pi/4$ | $\sqrt{2}/2$ | $\sqrt{2}/2$ | 1 |
| $\pi/3$ | $\sqrt{3}/2$ | $1/2$ | $\sqrt{3}$ |
| $\pi/2$ | 1 | 0 | $\infty$ |

| $x$ | $\log_e(x)$ | $\log_{10}(x)$ |
|---|---|---|
| 2 | 0.6931 | 0.3010 |
| 3 | 1.0986 | 0.4771 |
| 5 | 1.6094 | 0.6990 |
| 10 | 2.3026 | 1.0000 |

$$\begin{aligned}
\sin(a+b) &= \sin(a)\cos(b) + \cos(a)\sin(b) \\
\sin(a-b) &= \sin(a)\cos(b) - \cos(a)\sin(b) \\
\cos(a+b) &= \cos(a)\cos(b) - \cos(a)\sin(b) \\
\cos(a-b) &= \cos(a)\cos(b) + \cos(a)\sin(b) \\
\tan(a+b) &= \frac{\tan(a)+\tan(b)}{1-\tan(a)\tan(b)} \\
\tan(a-b) &= \frac{\tan(a)-\tan(b)}{1+\tan(a)\tan(b)}
\end{aligned}$$