

## 25 Further Robustness of the LQR

The most common robustness measures attributed to the LQR are a one-half gain reduction in any input channel, an infinite gain amplification in any input channel, or a phase error of plus or minus sixty degrees in any input channel. While these are general properties

(Continued on next page)

that have a clear graphical implication on the Bode or Nyquist plot, other useful robustness conditions can be developed. These include robustness to uncertainty in the real *coefficients* of the model (e.g., coefficients in the  $A$  matrix), and certain nonlinearities, including control switching and saturation. We will use the Lyapunov stability and the LMI formulation of matrix problems in this section to expand these ideas.

Saturation nonlinearities in particular are ubiquitous in control system application; we find them in "railed" conditions of amplifiers, rate and position limits in control surface actuators, and in well-meaning but ad hoc software limits. As shown below, moderate robustness in saturation follows from the basic analysis, but much stronger results can be obtained with new tools.

When the LQR is used to define the loop shape in the loop transfer recovery method (as opposed to the Kalman filter in the more common case), these robustness measures hold.

## 25.1 Tools

### 25.1.1 Lyapunov's Second Method

The idea of Lyapunov's Second Method is the following: if a positive definite function of the state  $\vec{x}$  can be found, with  $V(\vec{x}) = 0$  only when  $\vec{x} = \vec{0}$ , and if  $dV(\vec{x})/dt < 0$  for all time, then the system is stable. A useful analogy for the Lyapunov function  $V(\vec{x})$  is energy, dissipated in a passive mechanical system by damping, or in a passive electrical system through resistance.

### 25.1.2 Matrix Inequality Definition

Inequalities in the case of matrices are in the sense of positive and negative (semi) definiteness. Positive definite  $A$  means  $\vec{x}^T A \vec{x} > 0$  for all  $\vec{x}$ ; positive semidefinite  $A$  means  $\vec{x}^T A \vec{x} \geq 0$  for all  $\vec{x}$ . With  $A$  and  $B$  square and of the same dimension,

$$A < B \text{ means } \vec{x}^T A \vec{x} < \vec{x}^T B \vec{x}, \text{ for all } \vec{x}. \quad (300)$$

Also, we say for the case of a scalar  $\gamma$ ,

$$A < \gamma \text{ means } A - \gamma I < 0. \quad (301)$$

### 25.1.3 Franklin Inequality

A theorem we can use to assist in the Lyapunov analysis is the following, attributed to Franklin (1969).

$$A^T B + B^T A \leq \gamma A^T A + \frac{1}{\gamma} B^T B, \text{ for all real } \gamma > 0. \quad (302)$$

The scalar  $\gamma$  is a free parameter we can specify. It is assumed that the matrices  $A$  and  $B$  are of consistent dimensions.

### 25.1.4 Schur Complement

Consider the symmetric block matrix defined as follows:

$$M = \begin{bmatrix} A & B \\ B^T & D \end{bmatrix}. \quad (303)$$

The *Schur complements of blocks*  $A$  and  $D$  are defined, respectively, as

$$\begin{aligned} \Gamma_A &= D - B^T A^{-1} B \\ \Gamma_D &= A - B D^{-1} B^T, \end{aligned} \quad (304)$$

and an important property is that

$$M > 0 \iff \Gamma_A > 0 \text{ and } \Gamma_D > 0 \text{ and } A > 0 \text{ and } D > 0. \quad (305)$$

The Schur complements thus capture the sign of  $M$ , but in a smaller dimensioned matrix equation. The fact that both  $A$  and  $D$  have to be positive definite, independent of the Schur complements, is obvious by considering  $\vec{x}$  that involve only  $A$  or  $D$ , e.g.,  $\vec{x} = [\vec{1}_{1 \times n_A} \vec{0}_{1 \times n_D}]^T$ , where  $n_A$  and  $n_D$  are the dimensions of  $A$  and  $D$ .

### 25.1.5 Proof of Schur Complement Sign

It is easy to verify that

$$M = \begin{bmatrix} A & B \\ B^T & D \end{bmatrix} = \begin{bmatrix} I & 0 \\ B^T A^{-1} & I \end{bmatrix} \begin{bmatrix} A & 0 \\ 0 & D - B^T A^{-1} B \end{bmatrix} \begin{bmatrix} I & A^{-1} B \\ 0 & I \end{bmatrix} \quad (306)$$

The outer matrices are nonsingular so there is a one-to-one transformation

$$\vec{z} = \begin{bmatrix} I & A^{-1} B \\ 0 & I \end{bmatrix} \vec{x}, \quad (307)$$

and thus  $M > 0$  is equivalent to  $A > 0$  and  $\Gamma_A > 0$ . The proof for  $\Gamma_D$  is completely analogous.

### 25.1.6 Schur Complement of a Nine-Block Matrix

For the purposes of derivation below, consider now the symmetric matrix

$$M = \begin{bmatrix} A & B & C \\ B^T & D & 0 \\ C^T & 0 & E \end{bmatrix}. \quad (308)$$

Using the Schur complement of the diagonal block including  $D$  and  $E$ , we have

$$M > 0 \iff A - B D^{-1} B^T - C E^{-1} C^T > 0 \text{ and } D > 0 \text{ and } E > 0. \quad (309)$$

Again, positive definiteness of a large matrix is reflected in a smaller matrix inequality.

### 25.1.7 Quadratic Optimization with a Linear Constraint

Consider the quadratic function  $V = \vec{x}^T P \vec{x}$ ,  $P$  symmetric and positive definite, and the minimization of  $V$  subject to a constraint  $\vec{c}^T \vec{x} = a$ , where  $\vec{c}^T$  is a row vector, and  $a$  is a scalar. The solution is:

$$\min V = \frac{a^2}{\vec{c}^T P^{-1} \vec{c}}. \quad (310)$$

To show that this is true, augment the objective with a LaGrange multiplier to become

$$\tilde{V} = V + \lambda(\vec{c}^T \vec{x} - a). \quad (311)$$

Setting  $\partial \tilde{V} / \partial \vec{x} = 0$  gives  $2\vec{x}^T P + \lambda \vec{c}^T = 0$ . Post-multiply this equation by  $P^{-1} \vec{c}$ , and solve to give  $\lambda = -2a / \vec{c}^T P^{-1} \vec{c}$ . Then, using  $2\vec{x}^T P + \lambda \vec{c}^T = 0$  again, post-multiply by  $\vec{x} / 2$ , and insert  $\lambda$ , concluding the proof.

## 25.2 Comments on Linear Matrix Inequalities (LMI's)

The reason that solving  $M > 0$  is simpler than the equivalent Schur complement inequality is that, as long as  $M(\vec{y})$  is *affine* in the unknowns  $\vec{y}$ , the solution set is *convex*. Affine means that

$$M(\vec{y}) = M_o + K[\vec{y} \ \vec{y} \ \dots \ \vec{y}]N, \quad (312)$$

i.e., that  $M$  is a constant matrix plus terms linear in each element of  $\vec{y}$ . Note the matrices  $K$  and  $N$  here, as well as the columnwise expansion of  $\vec{y}$ , are needed to put elements of  $\vec{y}$  in arbitrary locations of  $M$ .

Convex means that if  $\vec{y}_1$  and  $\vec{y}_2$  are feasible solutions to the inequality, then so is  $\vec{y}_3 = \alpha \vec{y}_1 + (1 - \alpha) \vec{y}_2$ , for any  $\alpha$  between zero and one. A convex solution set has no hidden corners or shadows, because every point on a straight line between two solutions is also a solution. Computer programs solving convex optimization problems can always get the global optimum efficiently, or determine that no solution exists.

From the Schur complements above, clearly we can transform a (suitable) matrix equation that is quadratic or higher in the unknowns into an affine  $M(\vec{y})$ , and therefore make an efficient solution. The term *linear matrix inequality* refers of course to the fact that  $M$  is affine and hence linear in the unknown variables, but also to the methods of analysis and numerical solution, wherein the idea is to recognize

$$M(\vec{y}) = M_o + M_1 y_1 + M_2 y_2 + \dots M_n y_n, \quad (313)$$

where  $n$  is the dimension of the unknown vector  $\vec{y}$ .

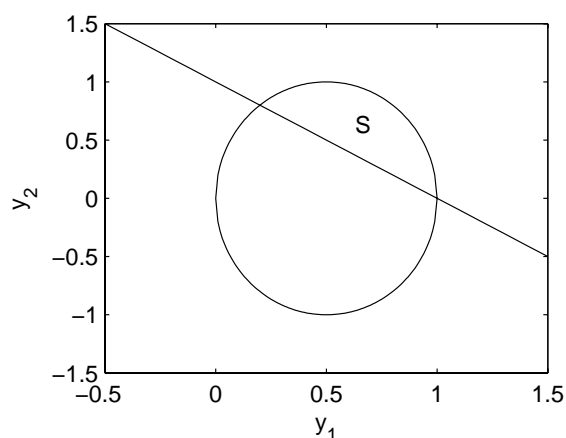
*Example.* Consider the LMI

$$M(\vec{y}) = \begin{bmatrix} y_1 - 1 & 2y_2 - 1 \\ y_2 + 1 & y_2 - y_1 \end{bmatrix} > 0. \quad (314)$$

We use the theorem that  $M > 0$  is achieved if each of the submatrices cornered at the (1,1) element has positive determinant. Then  $M > 0$  is equivalent to

$$\begin{aligned} y_1 &> 1 \\ (y_1 - 1)(y_2 - y_1) - (2y_2 - 1)(y_2 + 1) &> 0. \end{aligned} \quad (315)$$

The solution set is shown as  $S$  in the figure below;  $M$  is affine in  $\vec{y}$  and the solution set is convex.



## 25.3 Parametric Uncertainty in $A$ and $B$ Matrices

### 25.3.1 General Case

Let  $S$  be the solution to the matrix Riccati equation

$$A^T S + SA + Q - PBR^{-1}B^T P = 0. \quad (316)$$

We consider here perturbations to the design plant matrices  $A$  and  $B$  of the form  $\dot{\vec{x}} = (A + \Delta A)\vec{x} + (B + \Delta B)\vec{u}$ . The control in the LQR is  $\vec{u} = -R^{-1}B^T P\vec{x}$ . Select a Lyapunov function of the form  $V(\vec{x}) = \vec{x}^T S\vec{x}$ . Then

$$\begin{aligned} \dot{V} &= \dot{\vec{x}}^T S\vec{x} + \vec{x}^T S\dot{\vec{x}} \\ &= \vec{x}^T \left[ A^T S + SA + \Delta A^T S + S\Delta A - 2SBR^{-1}B^T S - \right. \\ &\quad \left. SBR^{-1}\Delta B^T S - S\Delta BR^{-1}BS \right] \vec{x}. \end{aligned} \quad (317)$$

We give the following special structures to the perturbations:

$$\begin{aligned}\Delta A &= D_A \Delta_A E_A \\ \Delta B &= D_B \Delta_B E_B.\end{aligned}\tag{318}$$

The idea is to capture all of the uncertainty in  $\Delta_A$  and  $\Delta_B$  and capture the structure of the perturbations using the other four matrices. We have, using the Franklin inequality,

$$\begin{aligned}\dot{V} &= \vec{x}^T \left[ A^T S + SA - 2SBR^{-1}B^T S + E_A^T \Delta_A^T D_A^T S + SD_A \Delta_A E_A \right. \\ &\quad \left. - SBR^{-1}E_B^T \Delta_B^T D_B^T S - SD_B \Delta_B E_B R^{-1}B^T S \right] \vec{x}\end{aligned}\tag{319}$$

$$\begin{aligned}&\leq \vec{x}^T \left[ A^T S + SA - 2SBR^{-1}B^T S + \gamma_A E_A^T \Delta_A^T \Delta_A E_A + \frac{1}{\gamma_A} SD_A D_A^T S \right. \\ &\quad \left. + \gamma_B SBR^{-1}E_B^T \Delta_B^T \Delta_B E_B R^{-1}B^T S + \frac{1}{\gamma_B} SD_B D_B^T S \right] \vec{x} \\ &\leq \vec{x}^T \left[ A^T S + SA - 2SBR^{-1}B^T S + \gamma_A E_A^T E_A + \frac{1}{\gamma_A} SD_A D_A^T S \right. \\ &\quad \left. + \gamma_B SBR^{-1}E_B^T E_B R^{-1}B^T S + \frac{1}{\gamma_B} SD_B D_B^T S \right] \vec{x},\end{aligned}\tag{320}$$

where the last inequality holds if

$$\Delta_A^T \Delta_A \leq I,$$

$$\Delta_B^T \Delta_B \leq I.\tag{321}$$

$$\tag{322}$$

For a given perturbation model with matrices  $D_A$ ,  $E_A$ ,  $D_B$ , and  $E_B$ , we thus obtain a matrix inequality which, if met, guarantees stability of the system for all  $\vec{x}$ :

$$\begin{aligned}A^T S + SA - 2SBR^{-1}B^T S + \gamma_A E_A^T E_A + \frac{1}{\gamma_A} SD_A D_A^T S \\ + \gamma_B SBR^{-1}E_B^T E_B R^{-1}B^T S + \frac{1}{\gamma_B} SD_B D_B^T S < 0.\end{aligned}\tag{323}$$

Since scalars  $\gamma_A$  and  $\gamma_B$  can take on any positive value, they remain to be picked, and in fact judicious choices are needed to maximize the robustness guarantees – i.e., to minimize the conservativeness.

### 25.3.2 Uncertainty in B

Putting aside  $\Delta A$  for the moment, and substituting in the Ricatti equation into the above leads to

$$-Q - SBR^{-1}B^T S + \gamma_B SBR^{-1}E_B^T E_B R^{-1}B^T S + \frac{1}{\gamma_B} S D_B D_B^T S < 0. \quad (324)$$

Following the presentation of gain and phase margins of the LQR in a previous section, we here consider gain margins through the diagonal matrix  $N$ , such that

$$\Delta_B = BN = B\sqrt{R^{-1}|N|} \times I \times \sqrt{R|N|} = D_B \Delta_B E_B. \quad (325)$$

$|N|$  denotes the matrix made up of the absolute values of the elements of  $N$ . This  $\Delta_B$  is a very specific structure, whose rationale becomes evident below. The stability inequality becomes

$$-Q - SBR^{-1}B^T S + \gamma_B SB|N|R^{-1}B^T S + \frac{1}{\gamma_B} SBR^{-1}|N|B^T S < 0, \quad (326)$$

or, equivalently,

$$I - \gamma_B |N| - \frac{1}{\gamma_B} |N| > 0, \quad (327)$$

because  $Q$  can be arbitrarily small in the LQR; certainly a large value of  $Q$  increases the robustness (see the case of  $\Delta A$  below, for example). Since  $N$  is diagonal, it is sufficient to keep all

$$1 - |N_{i,i}| \left( \gamma_B + \frac{1}{\gamma_B} \right) > 0. \quad (328)$$

Solving for  $\gamma_B$  gives

$$\gamma_B = \frac{1 \pm \sqrt{1 - 4N_{i,i}^2}}{2|N_{i,i}|}, \quad (329)$$

showing that real, positive  $\gamma_B$  are attained for  $|N_{i,i}| < 1/2$ . Hence, this analysis confirms the one-half gain reduction property derived earlier. On the other hand, it confers only a one-half gain amplification robustness, which is overly conservative. Usage of the Franklin inequality to symmetrize the components involving  $\Delta B$  is the cause of this, since it can be verified that using Equation 319 directly gives

$$\begin{aligned} -Q - SBR^{-1}(I + N + N)B^T S &< 0, \text{ or} \\ 1 + 2N_{i,i} &> 0, \end{aligned} \quad (330)$$

showing both the one-half reduction and infinite upward gain margins.

In summary, in the special case of diagonal  $N$  in the model of  $\Delta B$ , we crafted  $D_B$  and  $E_B$  so as to align terms of the form  $SBR^{-1}B^T S$ , simplifying the analysis. This leads to a conservative result using the Franklin inequality, but the full, standard result on a second look. Better results could be obtained for specific cases, but probably with greater effort.

### 25.3.3 Uncertainty in A

Uncertainty in the system matrix  $A$  has a different role in robustness than does uncertainty in  $B$ , which is primarily an issue of gain and does not affect the open-loop stability. Perturbations to the design matrix  $A$  cause the open- and closed-loop poles to move, potentially destabilizing either or both of the open- and closed-loop systems. Consider the main inequality for stability again, setting aside  $\Delta B$  now:

$$-Q - SBR^{-1}B^T S + \gamma_A E_A^T E_A + \frac{1}{\gamma_A} S D_A D_A^T S < 0. \quad (331)$$

Aligning the uncertain terms with  $SBR^{-1}B^T S$  (as with  $\Delta B$  above) proves to be impractical here because it imposes a specific structure on  $\Delta A$ , and requires that elements of  $\Delta A$  would have to change together. A more useful result involves a diagonal  $\Delta_A$ , and for the purposes of illustration, we consider a second order system as follows:

$$\Delta_A = D_A \Delta_A E_A = \begin{bmatrix} d_1 & d_2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \delta_1 & 0 \\ 0 & \delta_2 \end{bmatrix} I \quad (332)$$

$$= \begin{bmatrix} d_1 \delta_1 & d_2 \delta_2 \\ 0 & 0 \end{bmatrix}. \quad (333)$$

In addition, we set  $B = [b \ 0]^T$ , so that  $R$  is a scalar  $r$ , and we set  $Q$  diagonal. The resultant inequality is

$$-Q + \gamma_A I - S \left( \begin{bmatrix} b^2/r & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} (d_1^2 + d_2^2)/\gamma_A & 0 \\ 0 & 0 \end{bmatrix} \right) S < 0. \quad (334)$$

The condition for stability will be met if  $Q - \gamma_A I > 0$  and  $b^2/r - (d_1^2 + d_2^2)/\gamma_A > 0$ , or equivalently, setting  $\gamma_A = \min Q_{i,i}$ , if

$$\frac{b^2}{r} \min Q_{i,i} > d_1^2 + d_2^2. \quad (335)$$

Now let, for example,

$$\begin{aligned} A &= \begin{bmatrix} -0.5 & -0.5 \\ 1 & 0 \end{bmatrix}, \\ B &= [1 \ 0]^T \\ Q &= \text{diag}(1, 1) \\ R &= 1. \end{aligned}$$

This informs us that the robustness condition is  $d_1^2 + d_2^2 < 2$ , and very substantial perturbations in  $A$  are allowed, for example



$$A + \Delta A = \begin{bmatrix} 0.5 & 0.5 \\ 1 & 0 \end{bmatrix}.$$

Computed results that confirm the bound are shown in Figure 9. An interesting note is that the gain reduction margin in the presence of this  $\Delta A$  suffers dramatically, with failure of the robustness if  $b < 0.93$ . Intuitively, the uncertainty in  $A$  has used up almost all of the available safety margin.

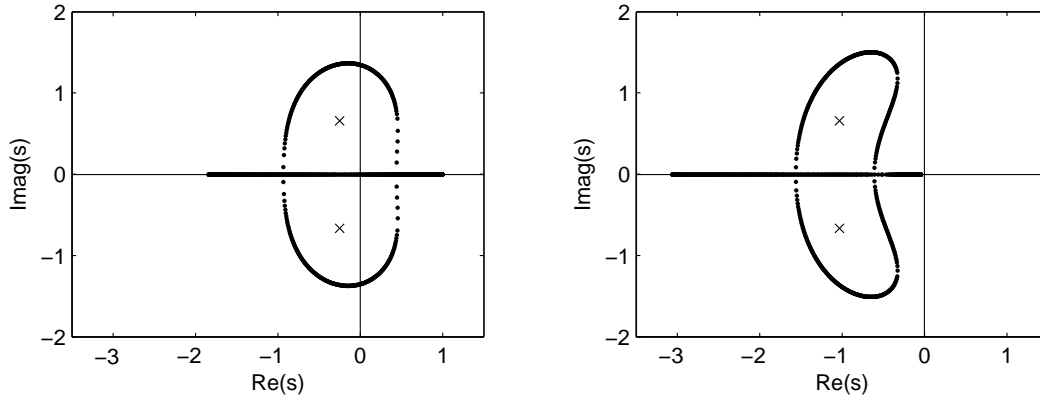


Figure 9: (left) open-loop pole locations along the boundary of the stability region in Equation 335; (right) closed-loop pole locations with LQR.

#### 25.3.4 $A$ and $B$ Perturbations as an LMI

Using the triple LMI of Equation 309, we have, for the combined  $A$  and  $B$  structured uncertainty models,

$$\begin{aligned} & -Q - SBR^{-1}B^T S + \gamma_A E_A^T E_A + \frac{1}{\gamma_A} S D_A D_A^T S \\ & + \gamma_B SBR^{-1}E_B^T E_B R^{-1}B^T S + \frac{1}{\gamma_B} S D_B D_B^T S < 0 \end{aligned} \quad (336)$$

is equivalent to

$$\begin{bmatrix} -Q - SBR^{-1}B^T S + \gamma_A E_A^T E_A + \gamma_B SBR^{-1}E_B^T E_B R^{-1}B^T S & S D_A & S D_B \\ D_A^T S & -\gamma_A I & 0 \\ D_B^T S & 0 & -\gamma_B I \end{bmatrix} < 0. \quad (337)$$

The solution set  $[\gamma_A \ \gamma_B]$ , if it exists, is convex. If a solution does not exist, one could lower the robustness by altering  $D_A$ ,  $E_A$ ,  $D_B$ , or  $E_B$ . Alternatively, increasing  $Q$ , the state penalty, where possible may increase robustness.

## 25.4 Input Nonlinearities

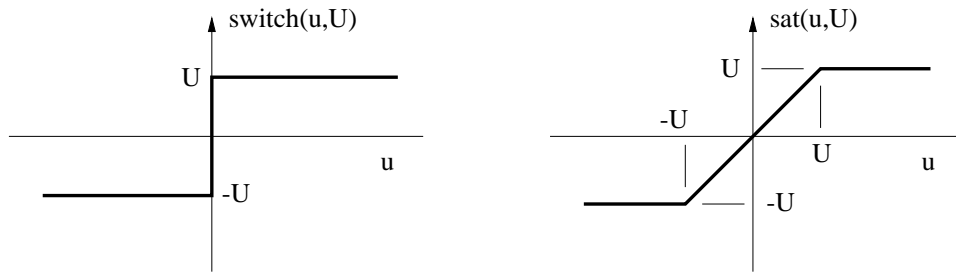
Control systems in practice often make use of switched or saturated action in the control channels. Switched and saturated control channels are defined respectively as

$$\text{switch}(u, U) = U \text{sign}(u), \quad (338)$$

and

$$\text{sat}(u, U) = U \text{sat}(u/U, 1) = \begin{cases} u, & |u| < U \\ U \text{sign}(u), & |u| > U. \end{cases}, \quad (339)$$

where the scalar  $U$  is positive. If the second argument of  $\text{sat}(\cdot)$  is omitted, it is taken to be one.



Our stability analysis so far is one wherein we seek to guarantee a negative rate in  $V(\vec{x})$  for all time. Referencing Equation 330, it is clear that *any*  $N$  can be tolerated within the range  $(-0.5, \infty)$ , *even if it is time varying*. It follows directly then that switching and saturated control channels cannot destabilize the LQR system until the gain reduction is more than one-half, that is,  $|u| > 2U$ . In practice, however, systems routinely operate beyond this point, and this is the aspect of saturated control in particular that we wish to pursue now. This section details the approach of Huang & Lam (2002), which uses LMI's and the structured uncertainty models above.

We say that  $f(\vec{u})$  is the diagonal function taking control commands from the LQR ( $\vec{u}$ ) to the input to the plant; elements of  $f$  correspond to the saturated operator above. Through scaling the  $B$  matrix, we can set  $U_i = 1$ , and hence  $f$ 's extreme outputs are  $\pm 1$ .

First, we assert that under the proper choice of positive definite matrix  $W$ , and for certain vectors  $\vec{z}$ , the following inequality holds:

$$2\vec{z}^T f(R^{-1}\vec{z}) \geq \vec{z}^T W R^{-1}\vec{z}. \quad (340)$$

If the control penalty matrix  $R$  is diagonal, and we constrain  $W$  also to be diagonal, this condition is met if, for each control channel,

$$2z_i \text{sat}\left(\frac{z_i}{R_{i,i}}\right) \geq z_i^2 \frac{W_{i,i}}{R_{i,i}}. \quad (341)$$

If the channel is not saturated, then we require  $W_{i,i} \leq 2$ . If the channel is saturated, then it is required instead that

$$2 \left| \frac{R_{i,i}}{z_i} \right| \geq W_{i,i}. \quad (342)$$

We will use the inequality of Equation 340 in our Lyapunov analysis, with special attention to these conditions. Because for the LQR we have  $z_i = e_i^T B^T S \vec{x}$ , where  $e_i$  is the unit vector in the  $i$ 'th direction, Equation 342 effectively places a bound on the state under saturated control, namely

$$|e_i^T B^T S \vec{x}| \leq 2R_{i,i}/W_{i,i}. \quad (343)$$

Viewed this way, clearly we would like to make each  $W_{i,i}$  as small as possible. As we show below, however, the matrix  $W$  appears in the only negative term of  $\dot{V}(\vec{x})$ ; the optimization then is to maximize the state space satisfying Equation 342, while keeping  $W$  big enough to guarantee stability.

With  $V(\vec{x}) = \vec{x}^T S \vec{x}$ , and considering uncertainty  $\Delta B$  only, we have

$$\dot{V}(\vec{x}) = \vec{x}^T (SA + A^T S) \vec{x} - 2\vec{x}^T SBf(R^{-1}B^T S \vec{x}) - 2\vec{x}^T S \Delta B f(R^{-1}B^T S \vec{x}). \quad (344)$$

Note that  $\Delta B$  never appears inside  $f$  because  $f$  operates on the control  $u$ , calculated with  $B$  alone. Applying the Franklin inequality to the last term yields

$$\begin{aligned} \dot{V}(\vec{x}) \leq & \vec{x}^T (SA + A^T S) \vec{x} - 2\vec{x}^T SBf(R^{-1}B^T S \vec{x}) \\ & + \gamma_B \vec{x}^T SD_B D_B^T S \vec{x} + \frac{1}{\gamma_B} f^T(R^{-1}B^T S \vec{x}) E_B^T E_B f(R^{-1}B^T S \vec{x}). \end{aligned} \quad (345)$$

Next, consider the remaining asymmetric term. Applying Equation 340, under the proper constraints on  $W$ , we obtain

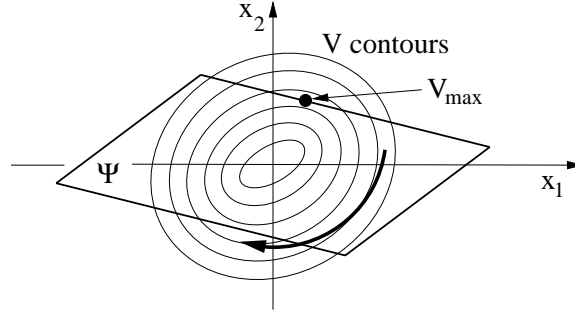
$$\begin{aligned} \dot{V}(\vec{x}) \leq & \vec{x}^T (SA + A^T S) \vec{x} - \vec{x}^T SBWR^{-1}B^T S \vec{x} \\ & + \gamma_B \vec{x}^T SD_B D_B^T S \vec{x} + \frac{1}{\gamma_B} f^T(R^{-1}B^T S \vec{x}) E_B^T E_B f(R^{-1}B^T S \vec{x}). \end{aligned} \quad (346)$$

Finally, the last term is bounded, giving

$$SA + A^T S - SBWR^{-1}B^T S + \gamma_B SD_B D_B^T S + \frac{1}{\gamma_B} SBR^{-1}E_B^T E_B R^{-1}B^T S < 0 \quad (347)$$

as the sufficient condition for stability. This last step is extremely conservative, since in fact  $|f_i| \leq 1$ ; Huang and Lam address this point.

The supporting conditions on elements of  $W$  are linear:  $W_{i,i}$  must be as in Equation 342, and less than two. As noted above, these translate to linear constraints on  $\vec{x}$ . Call the set of states that satisfy these conditions  $\Psi$ . The problem now is that although the Lyapunov function derivative is clearly less than zero for states within  $\Psi$ , the state trajectory could leave  $\Psi$ . This is entirely likely since  $V$  is a quadratic function, leading to elliptical contours as in the figure below.



As a result, the range of states leading to guaranteed stability is limited to those that are known not to leave  $\Psi$ . In other words, we have to settle for the largest ellipse completely contained in  $\Psi$ . We now invoke Equation 310, referencing Equation 343 as the set of linear constraints, to yield

$$V_{max} = \min_i \frac{4R_{i,i}^2}{W_{i,i}^2 e_i^T B^T S B e_i} \quad (348)$$

We would like to maximize  $V_{max}$ , which is equivalent to maximizing the minimum value of all the possible right hand sides of the above equation. This itself can be posed as an optimization in a new scalar  $k$ :

$$\text{minimize } k \text{ such that } kI - \frac{1}{2}R^{-1}W\Gamma > 0, \quad (349)$$

where  $\Gamma$  is diagonal, with  $\Gamma_{i,i} = \sqrt{e_i^T B^T S B e_i}$ . It follows that any  $\vec{x}$  is stable when  $V(\vec{x}) = \vec{x}^T S \vec{x} < 1/k^2$ .

In summary, we have three constraints and one optimization: Equation 347 for Lyapunov stability,  $0 \leq W_{i,i} \leq 2$  for cases where channels are not saturated, and Equation 349 for maximizing the level of  $V$  - an elliptical region in the state space - for which stability is guaranteed.

As in the case of no input saturation, this can be posed and solved as set of LMI's (which itself can be posed as a single LMI):

$$\begin{bmatrix} SA + A^T S - SBWR^{-1}B^T S + \gamma_B S D_B D_B^T S & SBR^{-1}E_B^T \\ E_B R^{-1}B^T S & \gamma_B I \end{bmatrix} < 0, \quad (350)$$

$$\frac{1}{2}R^{-1}W\Gamma < k, \quad (351)$$

$$W < 2, \quad (352)$$

$$0 < W. \quad (353)$$

To solve, pick a value for  $k$  and find a feasible solution. Decrease  $k$  as far as possible; the feasible solution achieves the maximum  $V$  from which the range of state space can be found from  $V = 1/k^2$ .