2 VESSEL INERTIAL DYNAMICS

We consider the rigid body dynamics with a coordinate system affixed on the body. A common frame for ships, submarines, and other marine vehicles has the body-referenced x-axis forward, y-axis to port (left), and z-axis up. This will be the sense of our body-referenced coordinate system here.

2.1 Momentum of a Particle

Since the body moves with respect to an inertial frame, dynamics expressed in the body-referenced frame need extra attention. First, linear momentum for a particle obeys the equality

$$\vec{F} = \frac{d}{dt}(m\vec{v}) \tag{19}$$

A rigid body consists of a large number of these small particles, which can be indexed. The summations we use below can be generalized to integrals quite easily. We have

$$\vec{F}_i + \vec{R}_i = \frac{d}{dt}(m_i\vec{v}_i), \tag{20}$$

where $\vec{F}_i$ is the external force acting on the particle and $\vec{R}_i$ is the net force exerted by all the other surrounding particles (internal forces). Since the collection of particles is not driven apart by the internal forces, we must have equal and opposite internal forces such that

(Continued on next page)

$$\sum_{i=1}^{N} \vec{R}_i = 0. \tag{21}$$

Then summing up all the particle momentum equations gives

$$\sum_{i=1}^{N} \vec{F}_i = \sum_{i=1}^{N} \frac{d}{dt} \left(m_i \vec{v}_i \right). \tag{22}$$

Note that the particle velocities are *not* independent, because the particles are rigidly attached.

Now consider a body reference frame, with origin **0**, in which the particle i resides at body-referenced radius vector $\vec{r}$; the body translates and rotates, and we now consider how the momentum equation depends on this motion.

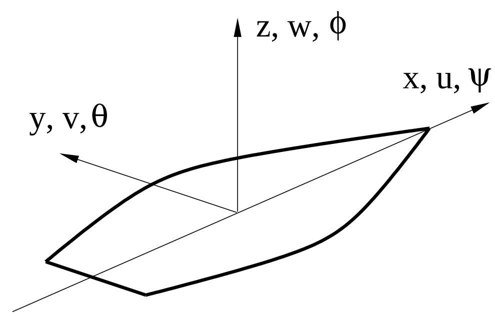

Figure 2: Convention for the body-referenced coordinate system on a vessel: x is forward, y is sway to the left, and z is heave upwards. Looking forward from the vessel bridge, roll about the x axis is positive counterclockwise, pitch about the y-axis is positive bow-down, and yaw about the z-axis is positive turning left.

2.2 Linear Momentum in a Moving Frame

The expression for total velocity may be inserted into the summed linear momentum equation to give

$$\begin{aligned} \sum_{i=1}^{N} \vec{F}_i &= \sum_{i=1}^{N} \frac{d}{dt}(m_i(\vec{v}_o + \vec{\omega} \times \vec{r}_i)) \\ &= m\frac{\partial \vec{v}_o}{\partial t} + \frac{d}{dt}\left[\vec{\omega} \times \sum_{i=1}^{N} m_i \vec{r}_i\right], \end{aligned} \tag{23}$$

where $m = \sum_{i=1}^{N} m_i$, and $\vec{v}_i = \vec{v}_o + \vec{\omega} \times \vec{r}_i$. Further defining the center of gravity vector $\vec{r}_G$ such that

$$m\vec{r}_G = \sum_{i=1}^{N} m_i \vec{r}_i, \tag{24}$$

we have

$$\sum_{i=1}^{N} \vec{F}_i = m\frac{\partial \vec{v}_o}{\partial t} + m\frac{d}{dt}(\vec{\omega} \times \vec{r}_G). \tag{25}$$

Using the expansion for total derivative again, the complete vector equation in body coordinates is

$$\vec{F} = \sum_{i=1} N = m\left(\frac{\partial \vec{v}_o}{\partial t} + \vec{\omega} \times \vec{v}_o + \frac{d\vec{\omega}}{dt} \times \vec{r}_G + \vec{\omega} \times (\vec{\omega} \times \vec{r}_G)\right). \tag{26}$$

Now we list some conventions that will be used from here on:

$$\begin{aligned}
\vec{v}_o &= \{u, v, w\} \text{ (body-referenced velocity)} \\
\vec{r}_G &= \{x_G, y_G, z_g\} \text{ (body-referenced location of center of mass)} \\
\vec{\omega} &= \{p, q, r\} \text{ (rotation vector, in body coordinates)} \\
\vec{F} &= \{X, Y, Z\} \text{ (external force, body coordinates).}
\end{aligned}$$

The last term in the previous equation simplifies using the vector triple product identity

$$\vec{\omega} \times (\vec{\omega} \times \vec{r}_G) = (\vec{\omega} \cdot \vec{r}_G)\vec{\omega} - (\vec{\omega} \cdot \vec{\omega})\vec{r}_G,$$

and the resulting three linear momentum equations are

$$\begin{aligned}
X &= m\left[\frac{\partial u}{\partial t} + qw - rv + \frac{dq}{dt}z_G - \frac{dr}{dt}y_G + (qy_G + rz_G)p - (q^2 + r^2)x_G\right] \\
Y &= m\left[\frac{\partial v}{\partial t} + ru - pw + \frac{dr}{dt}x_G - \frac{dp}{dt}z_G + (rz_G + px_G)q - (r^2 + p^2)y_G\right] \\
Z &= m\left[\frac{\partial w}{\partial t} + pv - qu + \frac{dp}{dt}y_G - \frac{dq}{dt}x_G + (px_G + qy_G)r - (p^2 + q^2)z_G\right].
\end{aligned} \tag{27}$$

Note that about half of the terms here are due to the mass center being in a different location than the reference frame origin, i.e., $\vec{r}_G \neq \vec{0}$.

2.3 Example: Mass on a String

Consider a mass on a string, being swung around around in a circle at speed U, with radius r. The centrifugal force can be computed in at least three different ways. The vector equation at the start is

$$\vec{F} = m\left(\frac{\partial \vec{v}_o}{\partial t} + \vec{\omega} \times \vec{v}_o + \frac{d\vec{\omega}}{dt} \times \vec{r}_G + \vec{\omega} \times (\vec{\omega} \times \vec{r}_G)\right).$$

2.3.1 Moving Frame Affixed to Mass

Affixing a reference frame *on* the mass, with the local x oriented forward and y inward towards the circle center, gives

$$\begin{aligned}
\vec{v}_o &= \{U, 0, 0\}^T \\
\vec{\omega} &= \{0, 0, U/r\}^T \\
\vec{r}_G &= \{0, 0, 0\}^T \\
\frac{\partial \vec{v}_o}{\partial t} &= \{0, 0, 0\}^T \\
\frac{\partial \vec{\omega}}{\partial t} &= \{0, 0, 0\}^T,
\end{aligned}$$

such that

$$\vec{F} = m\vec{\omega} \times \vec{v}_o = m\{0, U^2/r, 0\}^T.$$

The force of the string pulls in on the mass to create the circular motion.

2.3.2 Rotating Frame Attached to Pivot Point

Affixing the moving reference frame to the pivot point of the string, with the same orientation as above but allowing it to rotate with the string, we have

$$\begin{aligned}
\vec{v}_o &= \{0, 0, 0\}^T \\
\vec{\omega} &= \{0, 0, U/r\}^T \\
\vec{r}_G &= \{0, r, 0\}^T \\
\frac{\partial \vec{v}_o}{\partial t} &= \{0, 0, 0\}^T \\
\frac{\partial \vec{\omega}}{\partial t} &= \{0, 0, 0\}^T,
\end{aligned}$$

giving the same result:

$$\vec{F} = m\vec{\omega} \times (\vec{\omega} \times \vec{r}_G) = m\{0, U^2/r, 0\}^T.$$

2.3.3 Stationary Frame

A frame fixed in inertial space, and momentarily coincident with the frame on the mass (2.3.1), can also be used for the calculation. In this case, as the string travels through a small arc $\delta\psi$, vector subtraction gives

$$\delta \vec{v} \;=\; \{0, U \sin \delta\psi, 0\}^T \simeq \{0, U\delta\psi, 0\}^T.$$

Since $\dot{\psi} = U/r$, it follows easily that in the fixed frame $d\vec{v}/dt = \{0, U^2/r, 0\}^T$, as before.

2.4 Angular Momentum

For angular momentum, the summed particle equation is

$$\sum_{i=1}^{N}(\vec{M}_i + \vec{r}_i \times \vec{F}_i) = \sum_{i=1}^{N} \vec{r}_i \times \frac{d}{dt}(m_i \vec{v}_i), \tag{28}$$

where $\vec{M}_i$ is an external moment on the particle i. Similar to the case for linear momentum, summed internal moments cancel. We have

$$\begin{aligned}\sum_{i=1}^{N}(\vec{M}_i + \vec{r}_i \times \vec{F}_i) &= \sum_{i=1}^{N} m_i \vec{r}_i \times \left[\frac{\partial \vec{v}_o}{\partial t} + \vec{\omega} \times \vec{v}_o\right] + \sum_{i=1}^{N} m_i \vec{r}_i \times \left(\frac{\partial \vec{\omega}}{\partial t} \times \vec{r}_i\right) + \\ & \sum_{i=1}^{N} m_i \vec{r}_i \times (\vec{\omega} \times (\vec{\omega} \times \vec{r}_i)).\end{aligned}$$

The summation in the first term of the right-hand side is recognized simply as $m\vec{r}_G$, and the first term becomes

$$m\vec{r}_G \times \left[\frac{\partial \vec{v}_o}{\partial t} + \vec{\omega} \times \vec{v}_o\right]. \tag{29}$$

The second term expands as (using the triple product)

$$\begin{aligned}\sum_{i=1}^{N} m_i \vec{r}_i \times \left(\frac{\partial \vec{\omega}}{\partial t} \times \vec{r}_i\right) &= \sum_{i=1}^{N} m_i \left((\vec{r}_i \cdot \vec{r}_i)\frac{\partial \vec{\omega}}{\partial t} - \left(\frac{\partial \vec{\omega}}{\partial t} \cdot \vec{r}_i\right)\vec{r}_i\right) \\ &= \left\{\begin{array}{l} \sum_{i=1}^{N} m_i \left((y_i^2 + z_i^2)\dot{p} - (y_i \dot{q} + z_i \dot{r}) x_i\right) \\ \sum_{i=1}^{N} m_i \left((x_i^2 + z_i^2)\dot{q} - (x_i \dot{p} + z_i \dot{r}) y_i\right) \\ \sum_{i=1}^{N} m_i \left((x_i^2 + y_i^2)\dot{r} - (x_i \dot{p} + y_i \dot{q}) z_i\right) \end{array}\right\}.\end{aligned} \tag{30}$$

Employing the definitions of moments of inertia,

$$\begin{aligned}I &= \begin{bmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{bmatrix} \qquad \text{(inertia matrix)} \\ I_{xx} &= \sum_{i=1}^{N} m_i (y_i^2 + z_i^2) \\ I_{yy} &= \sum_{i=1}^{N} m_i (x_i^2 + z_i^2) \\ I_{zz} &= \sum_{i=1}^{N} m_i (x_i^2 + y_i^2) \\ I_{xy} &= I_{yx} = -\sum_{i=1}^{N} m_i x_i y_i \qquad \text{(cross-inertia)}\end{aligned}$$

$$\begin{aligned} I_{xz} &= I_{zx} = -\sum_{i=1}^{N} m_i x_i z_i \\ I_{yz} &= I_{zy} = -\sum_{i=1}^{N} m_i y_i z_i, \end{aligned}$$

the second term of the angular momentum right-hand side collapses neatly into $I\partial\vec{\omega}/\partial t$. The third term can be worked out along the same lines, but offers no similar condensation:

$$\begin{aligned} \sum_{i=1}^{N} m_i \vec{r}_i \times ((\vec{\omega}\cdot\vec{r}_i)\vec{\omega} - (\vec{\omega}\cdot\vec{\omega})\vec{r}_i) &= \sum_{i=1}^{N} m_i \vec{r}_i \times \vec{\omega}(\vec{\omega}\cdot\vec{r}_i) \\ &= \left\{ \begin{array}{l} \sum_{i=1}^{N} m_i(y_i r - z_i q)(x_i p + y_i q + z_i r) \\ \sum_{i=1}^{N} m_i(z_i p - x_i r)(x_i p + y_i q + z_i r) \\ \sum_{i=1}^{N} m_i(x_i q - y_i p)(x_i p + y_i q + z_i r) \end{array} \right\} \\ &= \left\{ \begin{array}{l} I_{yz}(q^2 - r^2) + I_{xz}pq - I_{xy}pr \\ I_{xz}(r^2 - p^2) + I_{xy}rq - I_{yz}pq \\ I_{xy}(p^2 - q^2) + I_{yz}pr - I_{xz}qr \end{array} \right\} + \\ & \quad \left\{ \begin{array}{l} (I_{zz} - I_{yy})rq \\ (I_{xx} - I_{zz})rp \\ (I_{yy} - I_{xx})qp \end{array} \right\}. \end{aligned} \tag{31}$$

Letting $\vec{M} = \{K, M, N\}$ be the total moment acting on the body, i.e., the left side of Equation 28, the complete moment equations are

$$\begin{aligned} K &= I_{xx}\dot{p} + I_{xy}\dot{q} + I_{xz}\dot{r} + \\ & \quad (I_{zz} - I_{yy})rq + I_{yz}(q^2 - r^2) + I_{xz}pq - I_{xy}pr + \\ & \quad m\left[y_G(\dot{w} + pv - qu) - z_G(\dot{v} + ru - pw)\right] \\ \\ M &= I_{yx}\dot{p} + I_{yy}\dot{q} + I_{yz}\dot{r} + \\ & \quad (I_{xx} - I_{zz})pr + I_{xz}(r^2 - p^2) + I_{xy}qr - I_{yz}qp + \\ & \quad m\left[z_G(\dot{u} + qw - rv) - x_G(\dot{w} + pv - qu)\right] \\ \\ N &= I_{zx}\dot{p} + I_{zy}\dot{q} + I_{zz}\dot{r} + \\ & \quad (I_{yy} - I_{xx})pq + I_{xy}(p^2 - q^2) + I_{yz}pr - I_{xz}qr + \\ & \quad m\left[x_G(\dot{v} + ru - pw) - y_G(\dot{u} + qw - rv)\right]. \end{aligned} \tag{32}$$

2.5 Example: Spinning Book

Consider a homogeneous rectangular block with $I_{xx} < I_{yy} < I_{zz}$ and all off-diagonal moments of inertia are zero. The linearized angular momentum equations, with no external forces or moments, are

$$I_{xx}\frac{dp}{dt} + (I_{zz} - I_{yy})rq = 0$$
$$I_{yy}\frac{dq}{dt} + (I_{xx} - I_{zz})pr = 0$$
$$I_{zz}\frac{dr}{dt} + (I_{yy} - I_{xx})qp = 0.$$

We consider in turn the stability of rotations about each of the main axes, with constant angular rate Ω. The interesting result is that rotations about the x and z axes are stable, while rotation about the y axis is not. This is easily demonstrated experimentally with a book or a tennis racket.

2.5.1 x-axis

In the case of the x-axis, $p = \Omega + \delta p$, $q = \delta q$, and $r = \delta r$, where the δ prefix indicates a small value compared to Ω. The first equation above is uncoupled from the others, and indicates no change in δp, since the small term $\delta q \delta r$ can be ignored. Differentiate the second equation to obtain

$$I_{yy}\frac{\partial^2 \delta q}{\partial t^2} + (I_{xx} - I_{zz})\Omega\frac{\partial \delta r}{\partial t} = 0$$

Substitution of this result into the third equation yields

$$I_{yy}I_{zz}\frac{\partial^2 \delta q}{\partial t^2} + (I_{xx} - I_{zz})(I_{xx} - I_{yy})\Omega^2 \delta q = 0.$$

A simpler expression is $\delta\ddot{q} + \alpha\delta q = 0$, which has response $\delta q(t) = \delta q(0)e^{\sqrt{-\alpha}t}$, when $\delta\dot{q}(0) = 0$. For spin about the x-axis, both coefficients of the differential equation are positive, and hence $\alpha > 0$. The imaginary exponent indicates that the solution is of the form $\delta q(t) = \delta q(0)cos\sqrt{\alpha}t$, that is, it oscillates but does not grow. Since the perturbation δr is coupled, it too oscillates.

2.5.2 y-axis

Now suppose $q = \Omega + \delta q$: differentiate the first equation and substitute into the third equation to obtain

$$I_{zz}I_{xx}\frac{\partial^2 \delta p}{\partial t^2} + (I_{yy} - I_{xx})(I_{yy} - I_{zz})\Omega^2 \delta p = 0.$$

Here the second coefficient has negative sign, and therefore $\alpha < 0$. The exponent is real now, and the solution grows without bound, following $\delta p(t) = \delta p(0)e^{\sqrt{-\alpha}t}$.

2.5.3 z-axis

Finally, let $r = \Omega + \delta r$: differentiate the first equation and substitute into the second equation to obtain

$$I_{yy} I_{xx} \frac{\partial^2 \delta p}{\partial t^2} + (I_{xx} - I_{zz})(I_{yy} - I_{zz})\Omega^2 \delta p = 0.$$

The coefficients are positive, so bounded oscillations occur.

2.6 Parallel Axis Theorem

Often, the mass center of an body is at a different location than a more convenient measurement point, the geometric center of a vessel for example. The parallel axis theorem allows one to translate the mass moments of inertia referenced to the mass center into another frame with parallel orientation, and vice versa. Sometimes a translation of coordinates to the mass center will make the cross-inertial terms I_{xy}, I_{yz}, I_{xz} small enough that they can be ignored; in this case $\vec{r}_G = \vec{0}$ also, so that the equations of motion are significantly reduced, as in the spinning book example.
The formulas are:

$$\begin{aligned} I_{xx} &= \bar{I}_{xx} + m(\delta y^2 + \delta z^2) \\ I_{yy} &= \bar{I}_{yy} + m(\delta x^2 + \delta z^2) \\ I_{zz} &= \bar{I}_{zz} + m(\delta x^2 + \delta y^2) \\ I_{yz} &= \bar{I}_{yz} - m\delta y \delta z \\ I_{xz} &= \bar{I}_{xz} - m\delta x \delta z \\ I_{xy} &= \bar{I}_{xy} - m\delta x \delta y, \end{aligned} \tag{33}$$

where $\bar{I}$ represents an MMOI in the axes of the mass center, and δx, for example, is the translation of the x-axis to the new frame. Note that translation of MMOI using the parallel axis theorem *must* be either to or from a frame resting exactly at the center of gravity.

2.7 Basis for Simulation

Except for external forces and moments $\vec{F}$ and $\vec{M}$, we now have the necessary terms for writing a full nonlinear simulation of a rigid body, in body coordinates. There are twelve states, comprising the following components:

- $\vec{v}_o$, the vector of body-referenced velocities.
- $\vec{\omega}$, body rotation rate vector.
- $\vec{x}$, location of the body origin, in *inertial* space.

- $\vec{E}$, Euler angle vector.

The derivatives of body-referenced velocity and rotation rate come from Equations 27 and 32, with some coupling which generally requires a 6×6 matrix inverse. The Cartesian position propagates according to

$$\dot{\vec{x}} = R^T(\vec{E})\vec{v}_o, \tag{34}$$

while the Euler angles follow:

$$\dot{\vec{E}} = \Gamma(\vec{E})\vec{\omega}. \tag{35}$$