2.094 — Finite Element Analysis of Solids and Fluids Fall '08

Lecture 14 - Total Lagrangian formulation, cont'd

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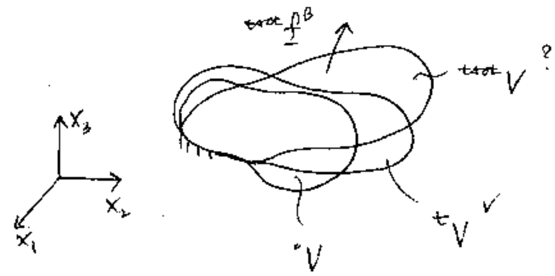
Truss element. 2D and 3D solids.



$$\int_{^{t+\Delta t}V} {}^{t+\Delta t}\tau_{ij}\delta_{t+\Delta t}e_{ij}d^{t+\Delta t}V = {}^{t+\Delta t}\mathcal{R} \tag{14.1}$$

$$\int_{^0V} {}^{t+\Delta t}_{\ \ 0}S_{ij}\delta^{t+\Delta t}_{\ \ 0}\epsilon_{ij}\delta^0V = {}^{t+\Delta t}\mathcal{R} \tag{14.2}$$

$$\Big\downarrow \text{linearization}$$

$$\int_{^0V} {}_0C_{ijrs\,0}e_{rs}\delta_0e_{ij}\delta^0V + \int_{^0V} {}^t_0S_{ij}\delta_0\eta_{ij}\delta^0V = {}^{t+\Delta t}\mathcal{R} - \int_{^0V} {}^t_0S_{ij}\delta_0e_{ij}\delta^0V \tag{14.3}$$

Note:

$$\delta_0e_{ij} = \delta^t_0\epsilon_{ij}$$

varying with respect to the configuration at time t.

F.E. discretization

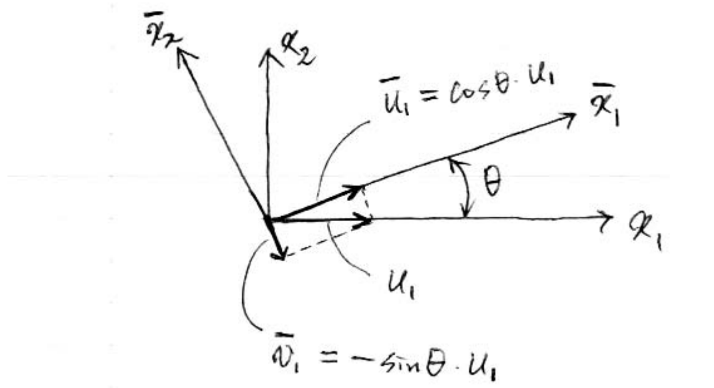
$$^0x_i = \sum_k h_k\,{}^0x_i^k \qquad {}^tx_i = \sum_k h_k\,{}^tx_i^k \qquad {}^{t+\Delta t}x_i = \sum_k h_k\,{}^{t+\Delta t}x_i^k \tag{14.4a}$$

$$^tu_i = \sum_k h_k\,{}^tu_i^k \qquad {}^{t+\Delta t}u_i = \sum_k h_k\,{}^{t+\Delta t}u_i^k \qquad u_i = \sum_k h_k u_i^k \tag{14.4b}$$

(14.4) into (14.3) gives

$$\left({}^t_0\boldsymbol{K}_L + {}^t_0\boldsymbol{K}_{NL}\right)\boldsymbol{U} = {}^{t+\Delta t}\boldsymbol{R} - {}^t_0\boldsymbol{F} \tag{14.5}$$

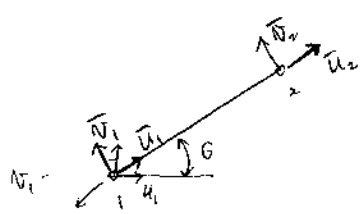
Truss



$\frac{\Delta L}{^0L} \ll 1$ small strain assumption:

$$
\begin{aligned}
{}^t_0\boldsymbol{K} &= \frac{E\,^0A}{^0L} \\
&= \begin{bmatrix} \cos^2\theta & \cos\theta\sin\theta & -\cos^2\theta & -\cos\theta\sin\theta \\ \cos\theta\sin\theta & \sin^2\theta & -\sin\theta\cos\theta & -\sin^2\theta \\ -\cos^2\theta & -\cos\theta\sin\theta & \cos^2\theta & \sin\theta\cos\theta \\ -\cos\theta\sin\theta & -\sin^2\theta & \sin\theta\cos\theta & \sin^2\theta \end{bmatrix} \\
&\quad + \frac{^tP}{^0L}\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}
\end{aligned}
\tag{14.6}
$$

(notice that the both matrices are symmetric)

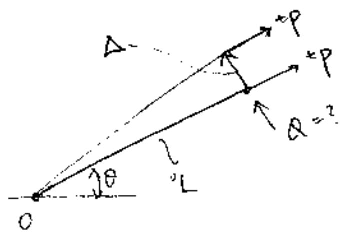


$$
\begin{pmatrix} \bar{u}_1 \\ \bar{v}_1 \end{pmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{pmatrix} u_1 \\ v_1 \end{pmatrix} \tag{14.7}
$$

Corresponding to the $\overline{u}$ and $\overline{v}$ displacements we have:

$$
{}^t_0\boldsymbol{K} = \frac{E\,^0A}{^0L} \tag{14.8}
$$

$$
= \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} + \frac{^tP}{^0L}\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} \tag{14.9}
$$



$$Q\,{}^0L = {}^tP \cdot \Delta \qquad \Rightarrow \qquad Q = \boxed{\frac{{}^tP}{{}^0L}} \cdot \Delta \tag{14.10}$$

where the boxed term is the stiffness. In axial direction, $\frac{{}^tP}{{}^0L}$ is not very important because usually $\frac{E\,{}^0A}{{}^0L} \gg \frac{{}^tP}{{}^0L}$. But, in vertical direction, $\frac{{}^tP}{{}^0L}$ is important.

$${}^t_0\boldsymbol{F} = {}^tP \begin{bmatrix} -\cos\theta \\ -\sin\theta \\ \cos\theta \\ \sin\theta \end{bmatrix} \tag{14.11}$$

2D/3D (e.g. Table 6.5) *2D:*

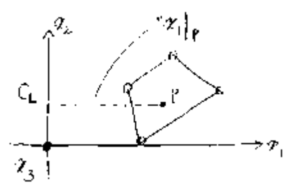
$${}_0\epsilon_{11} = \underbrace{{}_0u_{1,1} + {}^t_0u_{1,1}\,{}_0u_{1,1} + {}^t_0u_{2,1}\,{}_0u_{2,1}}_{{}_0e_{11}} + \frac{1}{2}\underbrace{\left[\left({}_0u_{1,1}\right)^2 + \left({}_0u_{2,1}\right)^2\right]}_{{}_0\eta_{11}} \tag{14.12}$$

$${}_0\epsilon_{22} = \cdots \tag{14.13}$$

$${}_0\epsilon_{12} = \cdots \tag{14.14}$$

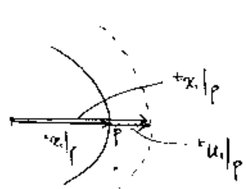
(Axisymmetric)

$${}_0\epsilon_{33} = \ ? \tag{14.15}$$



$${}^t_0\boldsymbol{\epsilon} = \frac{1}{2}\left(\left({}^t_0\boldsymbol{U}\right)^2 - \boldsymbol{I}\right) \tag{14.16}$$

$${}^t_0\boldsymbol{U}^2 = \begin{bmatrix} \times & \times & 0 \\ \times & \times & 0 \\ 0 & 0 & \underset{\substack{\uparrow \\ ({}^t\lambda)^2}}{\times} \end{bmatrix} \tag{14.17}$$



$$\begin{aligned} {}^t\lambda &= \frac{d\,{}^ts}{d\,{}^0s} = \frac{2\pi\left({}^0x_1 + {}^tu_1\right)}{2\pi\,{}^0x_1} \\ &= 1 + \frac{{}^tu_1}{{}^0x_1} \end{aligned} \tag{14.18}$$

$$\begin{aligned} {}^t_0\epsilon_{33} &= \frac{1}{2}\left[\left(1 + \frac{{}^tu_1}{{}^0x_1}\right)^2 - 1\right] \\ &= \frac{{}^tu_1}{{}^0x_1} + \frac{1}{2}\left(\frac{{}^tu_1}{{}^0x_1}\right)^2 \end{aligned} \tag{14.19}$$

$$ {}^{t+\Delta t}_{0}\epsilon_{33} = \frac{{}^{t}u_1 + u_1}{{}^{0}x_1} + \frac{1}{2}\left(\frac{{}^{t}u_1 + u_1}{{}^{0}x_1}\right)^2 \tag{14.20} $$

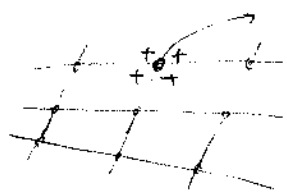
$$ {}_{0}\epsilon_{33} = {}^{t+\Delta t}_{0}\epsilon_{33} - {}^{t}_{0}\epsilon_{33} = \frac{u_1}{{}^{0}x_1} + \frac{{}^{t}u_1}{{}^{0}x_1} \cdot \frac{u_1}{{}^{0}x_1} + \frac{1}{2}\left(\frac{u_1}{{}^{0}x_1}\right)^2 \tag{14.21} $$

How do we assess the accuracy of an analysis?

Reading: Sec. 4.3.6

- Mathematical model $\sim \boldsymbol{u}$
- F.E. solution $\sim \boldsymbol{u}_h$

Find $\|\boldsymbol{u} - \boldsymbol{u}_h\|$ and $\|\boldsymbol{\tau} - \boldsymbol{\tau}_h\|$.



non-smoothed stress gives a good error measure.

References

[1] T. Sussman and K. J. Bathe. "Studies of Finite Element Procedures - on Mesh Selection." *Computers & Structures*, 21:257–264, 1985.

[2] T. Sussman and K. J. Bathe. "Studies of Finite Element Procedures - Stress Band Plots and the Evaluation of Finite Element Meshes." *Journal of Engineering Computations*, 3:178–191, 1986.