2.087
Differential Equations and Linear Algebras
MIT, Fall 2014

NAME:

Practice Final Exam Problems
Open Book, MATLAB® Allowed

9 problems are listed here. The real exam will also have 9 problems. But these problems a somewhat longer. The exam will last 90 minutes. To compare apples to apples, you might give yourself 120 minutes to do this practice set of problems.

Problem 1. Find the general solution to the following ODE

$$y'' + 4y' + 4y = f(t) \qquad y(0) = 0 \qquad y'(0) = 0 \qquad f(t) = \sin(2t)$$

Problem 2. Find the general solution to the following ODE:

$$(ty' - y)\cos\left(\frac{y}{t}\right) = t$$

Problem 3. Consider the following eight differential equations for $y(t)$. You do *not* need to solve these equations if you do not need the solution to answer the questions.

1. $y' + 2ty = e^{-t^2}\cos(8t)$
2. $yy' = y^2 - 4$
3. $y' = \sinh(y)$
4. $(ty' - y)\cos\left(\frac{y}{t}\right) = t$
5. $y'' + y' - 2y = 0$
6. $y'' - y' + 2y = 0$
7. $y'' + y = \frac{1}{t^2+1}$
8. $y'' + 4y' + 4y = f(t)$ where $f(t)$ is defined in Problem 1. $f(t) = \sin(2t)$

—

a. Which (if any) of these equations have periodically oscillating solutions as $t \to +\infty$?
b. Which (if any) of these equations have solutions that decay to zero equilibrium solution as $t \to +\infty$?
c. Which (if any) of these equations have solutions that blow up (in either finite or infinite time)?
d. Which (if any) of these equations have constant equilibrium solutions (either stable or unstable)?

aka initial conditions that would lead to constant solutions even in light of forcing if present

Problem 4. A plane takes off from location $(x, y) = (a, 0)$ due east of the destination airport at location $(x, y) = (0, 0)$. The cruising speed of the plane is $v_0 > 0$ and, additionally, wind constantly pushes the plane north (i.e. in the positive y direction) with a speed $w_0 > 0$. To deal with the wind and get to the destination airport, the flight plan of the pilot is to always point the plane in the direction of the destination airport.

This is not what a pilot would really do

Derive a differential equation that governs the path of the plane, that is, an equation containing *y*, *x*, and *dy*/*dx*

5) There are two matrices given by:

$$\mathbf{A} = \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} g & h \\ i & j \\ k & l \end{bmatrix}$$

For each question, either express the solution in terms of a, b, c, etc or else indicate "not defined"

a) What is the product $\mathbf{C}=\mathbf{AB}$?

b) What is the product $\mathbf{D}=\mathbf{B}^T\mathbf{A}^T$?

c) What is the product $\mathbf{E}=\mathbf{A}^T\mathbf{B}^T$?

d) What is the sum $\mathbf{F}= \mathbf{A}^T+\mathbf{B}$?

6) Solve the equation $\begin{bmatrix} 2 & -1 & 3 \\ 1 & -3 & -1 \\ 0 & 5 & 5 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 2 \\ -3 \\ 8 \end{bmatrix}$

If the equation has more than one solution, provide an expression for the whole family of solutions

7) Given a real n by n matrix $\mathbf{A}$ that is symmetric positive definite. Circle the single FALSE statement below

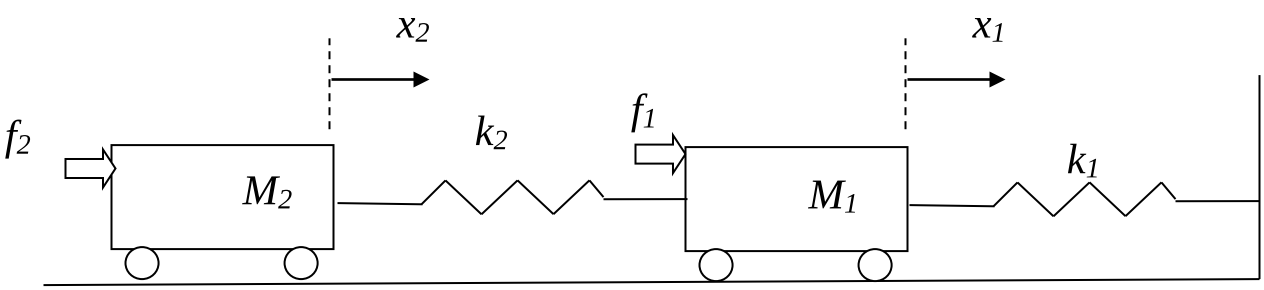
(a) $\mathbf{u}^T\mathbf{A}\mathbf{u}$ is a non-zero scalar for every $\mathbf{u}$ except when $\mathbf{u}$ is an n by 1 vector of zeros

(b) All eigenvalues of $\mathbf{A}$ are real and positive

(c) $\mathbf{A}^T = \mathbf{A}^{-1}$

(d) For any given n x1 vector $\mathbf{f}$, the equation $\mathbf{A}\mathbf{u} = \mathbf{f}$ has a unique solution

8. In the system depicted below, the carts roll horizontally with negligible loss due to air drag or rolling resistance of their wheels. The carts are connected by springs well modeled by Hooke's law.



a) Write a stiffness matrix so that **f**=**Kx** where **f** is a column vector of forces applied to the carts and **x** is a column vector of resulting displacements of the carts
b) Write the equations of motion. You may want to write them in state space form also.
c) Consider the case k_1=20N/m, k_2=30N/m, M_1=10kg, M_2=7kg. Given initial conditions x_1=1cm, x_2=0cm, find a solution to the equation from (b).
d) Find an initial condition resulting in the lowest maximum acceleration of the cart M_1 consistent with at least 1cm of total displacement of cart M_2

9)

a) Find the unit impulse response $g(t)$ of a system which is represented by the equation $2y'' + 8y' + 6y = \delta(t)$

b) Find the response of the system to a forcing given below

$$2y'' + 8y' + 6y = f(t) \qquad f(t) = \begin{cases} 0 & t < 0 \\ 1 & 0 \le t < 7 \\ -1 & 7 \le t < 9 \\ 0 & t > 9 \end{cases}$$

NOTE: $y = \int_0^t g(t-s)\,f(s)\,ds$