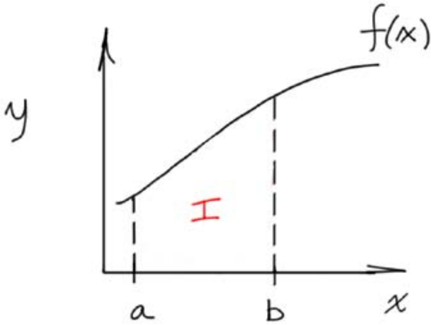
Integration

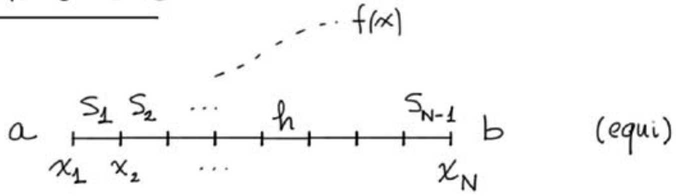
Introduction



$$I = \int_a^b f(x)\,dx$$

from Interpolation
to Integration

Formulation



$$I \equiv \int_a^b f(x)\,dx = \sum_{i=1}^{N-1} \int_{S_i} f(x)\,dx$$

difficult

$$\approx \sum_{i=1}^{N-1} \int_{S_i} (\mathcal{I}f)(x)\,dx \equiv I_h$$

easy

Error Analysis (general)

$$|I - I_h| = \left| \sum_{i=1}^{N-1} \int_{S_i} f(x)\,dx - \sum_{i=1}^{N-1} \int_{S_i} (\mathcal{I}f)(x)\,dx \right|$$

$$= \left| \sum_{i=1}^{N-1} \int_{S_i} \big(f(x) - (\mathcal{I}f)(x) \big)\,dx \right|$$

$$\leq \sum_{i=1}^{N-1} \left| \int_{S_i} \big(f(x) - (\mathcal{I}f)(x) \big)\,dx \right|$$

$$\leq \sum_{i=1}^{N-1} \int_{S_i} | f(x) - (\mathcal{I}f)(x) |\,dx$$

$$\leq \sum_{i=1}^{N-1} \int_{S_i} | f(x) - (\mathcal{I}f)(x) |\,dx$$

$$\leq \sum_{i=1}^{N-1} \max_{x \text{ in } S_i} | f(x) - (\mathcal{I}f)(x) | \int_{S_i} dx$$

interpolant error

$$\leq \sum_{i=1}^{N-1} e_{max}\, h \qquad h = \frac{b-a}{N-1}$$

$$\leq (N-1)\, h\, e_{max} = (b-a)\, e_{max}$$

$$\leq (b-a)\, C\, h^{p_{\mathcal{I}}}$$

may be pessimistic

order of interpolation scheme

$\mathcal{I}$: Piecewise-Constant, Left

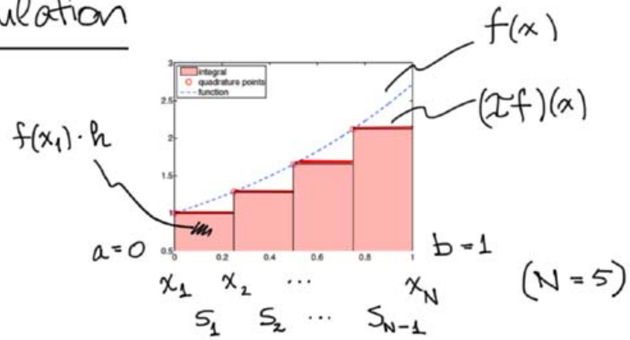
$p^{\mathcal{I}} = 1 : e_{max} = O(h^1)$

$\Downarrow$

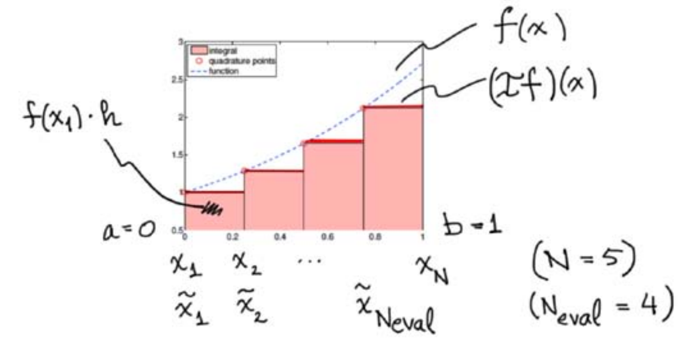
I_h: Rectangle Rule, Left

expect $|I - I_h| = O(h^1)$

Formulation



$$I_h = \sum_{i=1}^{N-1} \int_{S_i} (\mathcal{I}f)(x)\,dx = \sum_{i=1}^{N-1} f(x_i) \cdot h$$



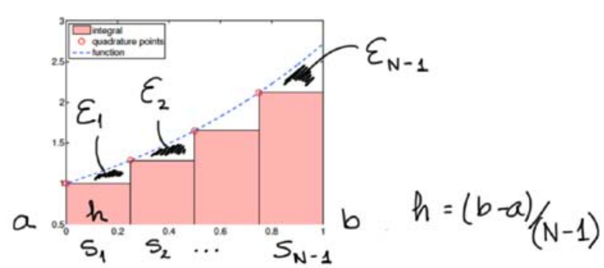
$$I_h = \sum_{i=1}^{N-1} \int_{S_i} (\mathcal{I}f)(x)\,dx = \sum_{i=1}^{N-1} h \cdot f(x_i)$$

$$= \sum_{i=1}^{N-1} w_i\, f(\tilde{x}_i)$$

quadrature points

quadrature weights

Error Analysis

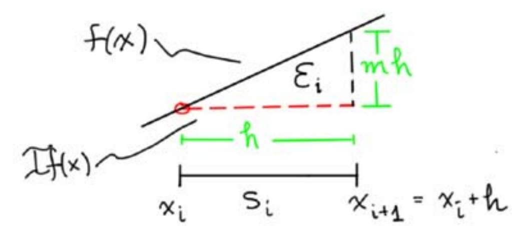


$h = (b-a)/(N-1)$

$$|I - I_h| \le \sum_{i=1}^{N-1} \mathcal{E}_i \le \sum_{i=1}^{N-1} \mathcal{E}_{max} = (N-1)\,\mathcal{E}_{max}$$

$$= \frac{b-a}{h}\,\mathcal{E}_{max}$$

say $f(x) = mx + c$ $\qquad m = f'(x)$



$$\mathcal{E}_i = \frac{1}{2}|m|\,h^2 \Rightarrow \mathcal{E}_{max} = \frac{1}{2}|m|\,h^2$$

$$\Downarrow$$

$$|I - I_h| \le \frac{(b-a)}{h}\,\frac{1}{2}|m|\,h^2$$

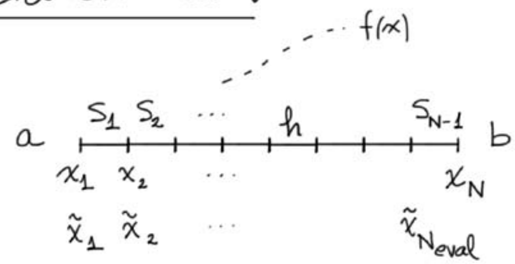
$$= (b-a)\,\frac{1}{2}\max_{a \le x \le b}|f'(x)|\,h$$

first order

general, rigorous result

DEMO

Operation Count



(equi)

$N_{eval} = N-1$

$$I_h = \sum_{i=1}^{N_{eval}} h \cdot f(\tilde{x}_i)$$

(i) function evaluations: $x_i \to f(\tilde{x}_i)$, $1 \le i \le N_{eval}$

(ii) sum: $O(N_{eval})$ FLOPs

$\mathcal{I}$: Piecewise-Linear

$p^{\mathcal{I}} = 2$: $e_{max} = O(h^2)$

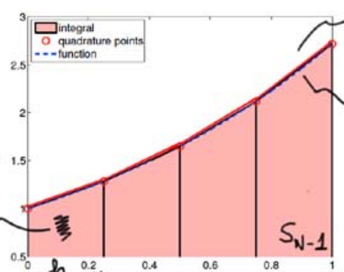
$\Downarrow$

I_h: Trapezoidal Rule

expect $|I - I_h| = O(h^2)$

Formulation

trapezoid area:
$h \cdot \frac{1}{2}(f(x_1) + f(x_2))$



$\mathcal{I}f(x)$

$f(x)$

x_1 h x_2 . . . x_N

$\tilde{x}_1$ $\tilde{x}_2$ $\tilde{x}_{N_{eval}}$

$N_{eval} = N$

$$I_h = \sum_{i=1}^{N-1} \underbrace{h}_{h_i} \frac{1}{2}\left(f(x_i) + f(x_{i+1})\right) = \sum_{i=1}^{N_{eval}} w_i f(\tilde{x}_i)$$

$w_1 = \frac{h}{2}$, $\underset{2 \le i \le N_{eval}-1}{w_i = h}$, $w_{N_{eval}} = \frac{h}{2}$

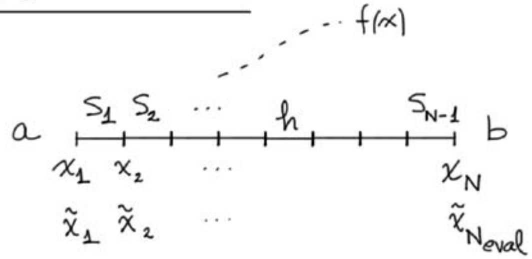
Error Analysis

$$|I - I_h| \le (b-a)\frac{h^2}{12} \max_{a \le x \le b} |f''(x)|$$

$\Downarrow$

$|I - I_h| = O(h^2)$ as expected

Operation Count



(equi)

$N_{eval} = N$

$$I_h = \frac{h}{2} \cdot f(\tilde{x}_1) + \sum_{i=2}^{N_{eval}-1} h \cdot f(\tilde{x}_i) + \frac{h}{2} \cdot f(\tilde{x}_{N_{eval}})$$

(i) function evaluations: $\tilde{x}_i \to f(\tilde{x}_i)$, $1 \le i \le N_{eval}$

(ii) sum: $O(N_{eval})$ FLOPs

Variable h: h_i, $1 \le i \le N_{eval} - 1$

$$I_h = \sum_{i=1}^{N_{eval}-1} h_i \, \frac{1}{2}\left(f(\tilde{x}_i) + f(\tilde{x}_{i+1})\right)$$

or

$$I_h = \sum_{i=1}^{N_{eval}} w_i \, f(\tilde{x}_i)$$

$$w_1 = \frac{1}{2} h_1, \quad w_i = \frac{1}{2}(h_{i-1} + h_i), \quad w_{N_{eval}} = \frac{1}{2} h_{N_{eval}-1}$$
$$2 \le i \le N_{eval} - 1$$

Perspectives

Are

some interpolants better than other interpolants,

$\downarrow$

some quadrature points special

such that

we improve upon $|I - I_h| = O(h^{p_I})$?

What if

- $f(x)$ is not smooth? $\quad f, f', f'' \ldots$
- $f(x)$ undergoes rapid variation? $\quad f', f'', \ldots$
- we wish to consider higher-order interpolants,

  $(\mathcal{I}f)(x)$: piecewise-quadratic, -cubic, ...

  to generate our "quadrature rule"?
- we wish to estimate the error $|I - I_h|$?

What if

- we wish to incorporate derivative conditions,

  $$(\mathcal{I}f)'(x_i) = f'(x_i), \ldots$$

  to generate our quadrature rule?
- evaluation of $f(x)$ is not exact:

  $$x \rightarrow f(x) + \begin{array}{l} \text{error} \sim \text{FP arithmetic}, \ldots \\ \quad \text{or} \\ \text{noise} \sim \text{measurement}, \ldots \end{array} ?$$