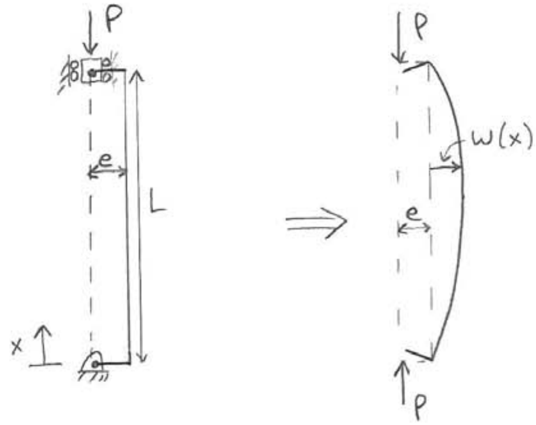
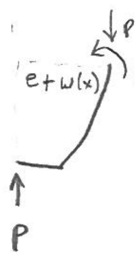
Recitation 8

Example: Pin-pin supported column with eccentric axial load.



External Bending Moment



$M(x) = P(e + w(x))$

Internal Bending Moment

$M = EI\kappa = -EIw''$

$$EIw'' + P(e + w)$$

$$(\text{Let } k^2 = \frac{P}{EI})$$

$$\boxed{w'' + k^2 w = -k^2 e}$$

General Solution

$w = w_{\text{h}} + w_{\text{p}}$

$w_{\text{h}} = C_1 \sin kx + C_2 \cos kx$

$w_{\text{p}} = -e$

So $w(x) = C_1 \sin kx + C_2 \cos kx - e$

BC's

$w(0) = 0$: $C_1(0) + C_2(1) - e = 0 \rightarrow C_2 = e$

$w(L) = 0$: $C_1 \sin(kL) + e\cos(kL) - e = 0 \rightarrow C_1 = \dfrac{e(1-\cos(kL))}{\sin(kL)}$

$$C_1 = e\tan(\frac{kL}{2})$$

So $\boxed{w(x) = e\left(\tan\left(\frac{kL}{2}\right)\sin(kx) + \cos(kx) - 1\right)}$

(<u>Note</u>: This is a *bending* problem thus far, valid for any P.)

<u>Max. deflection</u> (w_o) is at $x = L/2$:

$$w_o = e\left(\tan\frac{kL}{2}\sin\frac{kL}{2} + \cos\frac{kL}{2} - 1\right)$$

$$\boxed{w_o = e\left(\sec\frac{kL}{2} - 1\right)}$$

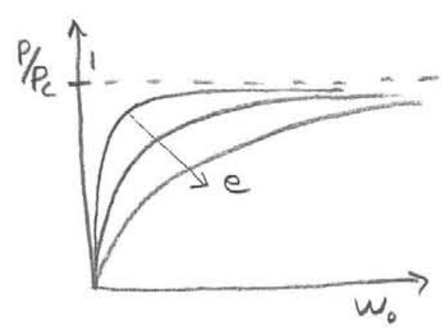
$$\begin{aligned}
&\tan x \sin x + \cos x \\
=&\frac{\sin^2 x}{\cos x} + \cos x\frac{\cos x}{\cos x} \\
=&\frac{\sin^2 x + \cos^2 x}{\cos x} \\
=&\frac{1}{\cos x} = \sec x
\end{aligned}$$

Now introduce the buckling load without eccentricity: $P_c = \dfrac{\pi^2 EI}{L^2}$

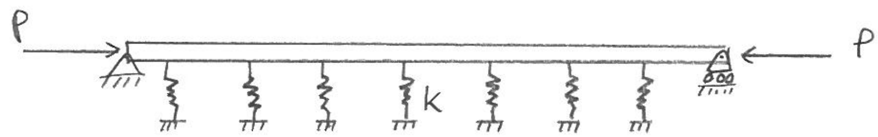
$$k^2 = \frac{P}{EI} \rightarrow \frac{kL}{2} = \frac{L}{2}\sqrt{\frac{P}{EI}} = \frac{L}{2}\sqrt{\frac{P}{P_c}\cdot\frac{P_c}{EI}} = \frac{L}{2}\sqrt{\frac{P}{P_c}\cdot\frac{\pi^2}{L^2}} = \frac{\pi}{2}\sqrt{\frac{P}{P_c}}$$

$$\frac{kL}{2} = \frac{\pi}{2}\sqrt{\frac{P}{P_c}}$$

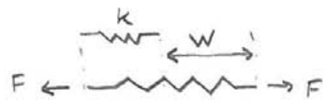
So $\boxed{w_o = e\left[\sec\left(\frac{\pi}{2}\sqrt{\frac{P}{P_c}}\right) - 1\right]}$



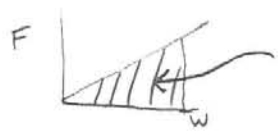
Example: Column on an elastic foundation



Linear elastic spring:



$$F = kw$$



$$\text{Work} = \int F\mathrm{d}w = \frac{1}{2}Fw = \frac{1}{2}kw^2$$

$$U = \int_0^L \left(\frac{1}{2}EI(w'')^2 + \frac{1}{2}P\epsilon + \frac{1}{2}kw^2\right)\mathrm{d}x$$

$\nearrow$ bending; $\uparrow$ axial compression; $\nwarrow$ spring foundation

$\Pi = U - Pu_o$
Apply Trefftz condition: $\delta^2\Pi = 0$

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$\delta^2\Pi = EI\int_0^L \delta w''\delta w''\mathrm{d}x + k\int_0^L \delta w\delta w\mathrm{d}x - P\int_0^L \delta w'\delta w'\mathrm{d}x = 0$

$$\Rightarrow P_{\mathrm{cr}} = \frac{\int_0^L EI(\delta w'')^2\mathrm{d}x + k\int_0^L(\delta w)^2\mathrm{d}x}{\int_0^L(\delta w')^2\mathrm{d}x}$$

Rayleigh-Ritz quotient: $$\boxed{P_{\mathrm{cr}} = \frac{EI\int_0^L \Phi''\Phi''\mathrm{d}x + k\int_0^L \Phi\Phi\mathrm{d}x}{\int_0^L \Phi'\Phi'\mathrm{d}x}}$$

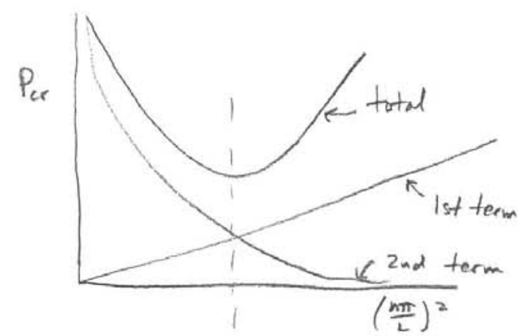
For pin-pin column, assume

$$\Phi = \sin\frac{n\pi x}{L}$$
$$\Phi' = \frac{n\pi}{L}\cos\frac{n\pi x}{L}$$
$$\Phi'' = \left(\frac{n\pi}{L}\right)^2\sin\frac{n\pi x}{L}$$

$$P_{\text{cr}} = \frac{EI\int_0^L \left(\frac{n\pi}{L}\right)^4 \sin^2\frac{n\pi x}{L}\mathrm{d}x + k\int_0^L \sin^2\frac{n\pi x}{L}\mathrm{d}x}{\int_0^L \left(\frac{n\pi}{L}\right)^2 \cos^2\frac{n\pi x}{L}\mathrm{d}x}$$

$$= \frac{EI\left(\frac{n\pi}{L}\right)^4 + k}{\left(\frac{n\pi}{L}\right)^2} \qquad \left(\text{because } \int_0^L \sin^2\frac{n\pi x}{L} = \int_0^L \cos^2\frac{n\pi x}{L}\right)$$

$$\boxed{P_{\text{cr}} = EI\left(\frac{n\pi}{L}\right)^2 + \frac{k}{\left(\frac{n\pi}{L}\right)^2}}$$



Find $P_{\text{cr,min}}$: (Let $x = (\frac{n\pi}{L})^2$)

$$\frac{\mathrm{d}P_{\text{cr}}}{\mathrm{d}x} = EI - \frac{k}{x^2} = 0$$

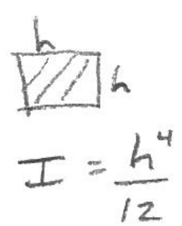
$$x^2 = \frac{k}{EI} \to x = \left(\frac{n\pi}{L}\right)^2 = \sqrt{\frac{n\pi}{L}}$$

So $P_{\text{cr,min}} = EI\sqrt{\frac{k}{EI}} + \frac{k}{\sqrt{\frac{k}{EI}}} = \sqrt{EIk} + \sqrt{EIk} = \boxed{2\sqrt{EIk}}$

Independent of L!

$$\left(\frac{n\pi}{L}\right)^2 = \sqrt{\frac{k}{EI}} \to n = \frac{L}{\pi}\left(\frac{k}{EI}\right)^{1/4}$$

Assume $h \times h$ cross-section, $L/h = 30$:



Find k_{cr} such that $n = 1$:

$$1 = \frac{L}{\pi}\left(\frac{k}{E\frac{h^4}{12}}\right)^{1/4} = \frac{L}{h}\frac{(12)^{1/4}}{\pi}\left(\frac{k}{E}\right)^{1/4} \simeq 0.6\frac{L}{h}\left(\frac{k}{E}\right)^{1/4}$$

$$\left(\frac{k}{E}\right)^{1/4} = \frac{1}{.6(30)} \;\to\; k = \left(\frac{1}{.6(30)}\right)^4 E \simeq 10^{-5}E$$

$k \leqslant k_{\mathrm{cr}}$:

$k > k_{\mathrm{cr}}$: