

Lecture 10: Advanced Topic in Column Buckling

10.1 The Tallest Column

In 1757 the Swiss mathematician Leonard Euler presented the famous solution for buckling of a pin-pin column under compressive loading at its end. He also formulated and solved the much more difficult problem of a clamped-free column loaded by its own weight. The practical question was how tall the prismatic column could be before it buckles under its own weight. In order to formulate this problem, the equation of equilibrium of a beam/column in the axial direction must be re-visited. Instead the equation $N' = 0$ or $N = \text{const}$, we must assume that there is a body force q per unit length $q = A\rho$, where A is the cross-sectional area of the column and ρ is its mass density. Then, the equilibrium in the axial direction requires that

$$N' = q \quad \text{or} \quad N = qx + C \quad (10.1)$$

In the coordinate system shown in Fig. (10.1), the axial force must be zero at $x = l$.

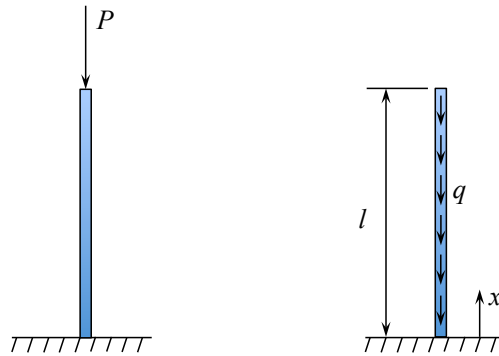


Figure 10.1: Column loaded at its tip (left) and loaded by its own weight (right).

The distribution of axial force along the length of the column is

$$N(x) = -q(l - x) \quad (10.2)$$

where the minus sign indicates that N is the compressive force. As before, the input parameters of the problem are E , I and q and the unknown is the critical length l_c .

The derivation of the buckling problem for a classical column presented in Lecture 9 is still valid but the axial force in Eq. (9.20) is no longer constant and thus should be kept inside the integral.

For the present problem the first variation of the total potential energy is

$$\delta\Pi = - \int_0^l M \delta w'' dx + \int_0^l q(l - x) w' \delta w' dx \quad (10.3)$$

Integrating the right hand side of Eq. (10.3) by part, one gets

$$\int_0^l [M'' + q(l-x)w'] \delta w \, dx + \text{Boundary terms} = 0 \quad (10.4)$$

where

$$\text{Boundary terms} = -M\delta w'|_0^l + M'\delta w|_0^l + q(l-x)w'\delta w \quad (10.5)$$

at $x = 0$, $\delta w = \delta w' = 0$; and at $x = l$, $M = 0$, $V = M' = 0$ and $l - x = 0$. Therefore the boundary terms vanish (see the dedication in Section 3.5). Using the elasticity law, $M = -EIw''$, the local equilibrium equation for the column self buckling becomes

$$EI \frac{d^4 w}{dx^4} + \frac{d}{dx} \left[q(l-x) \frac{dw}{dx} \right] = 0 \quad (10.6)$$

Integrating once, we get

$$EI \frac{d^3 w}{dx^3} + q(l-x)dw = 0 \quad (10.7)$$

The integration constant is zero because the shear force vanishes at the free end $x = l$. The governing equation is the third order linear differential equation with a variable coefficient. The solution is no longer represented by the harmonic function. The way to solve the problem is to introduce two new variables

$$\xi = \frac{2}{3} \sqrt{\frac{q(l-x)^3}{EI}}, \quad u = \frac{dw}{d\xi} \quad (10.8)$$

Then, Eq. (10.7) transforms to the Bessel equation

$$\frac{d^2 u}{d\xi^2} + \frac{1}{\xi} \frac{du}{d\xi} + \left(1 - \frac{1}{9\xi^2}\right)u = 0 \quad (10.9)$$

Omitting the details of the calculation, the critical length of the column is found to be

$$l_c^3 = \frac{9EI}{4q} j_{\frac{1}{3}}^2 \quad (10.10)$$

where $j_{\frac{1}{3}} = 1.866$ is the root of the Bessel function of the third kind. Finally

$$l_c^3 = 7.837 \frac{EI}{q} \quad (10.11)$$

The total weight of the column material is $N_c = l_c q$. In terms of the total weight, the critical length is

$$l_c^2 = 7.84 \frac{EI}{N_c} \quad (10.12)$$

For comparison, the length of the free-clamped column at buckling loaded by the same weight is

$$l_c^2 = \frac{\pi^2}{4} \frac{EI}{N_c} = 2.47 \frac{EI}{N_c} \quad (10.13)$$

The bottom of both column sees the same weight, but the critical length of the column undergoing self-buckling is $\sqrt{\frac{7.84}{2.47}} = 1.78$ times taller than a similar cross-section column loaded at its tip.

Example

A steel tubular mast solidly built-in the foundation and is free on its top. The cylinder is $t = 3$ mm thick and has a radius of $R = 50$ mm. What is the critical length of the mast to buckle under its own weight?

The total weight of the mast is

$$N_c = Al\rho \quad (10.14)$$

where A is the cross-sectional area, $A = 2\pi Rt$. The second moment of inertia of the thin-walled tube is $I = \pi R^3 t$. From Eq. (10.12)

$$l_c^2 = 7.84 \frac{E\pi R^3 t}{2\pi R t l_c \rho} \quad (10.15)$$

from which one gets

$$l_c = \sqrt[3]{\frac{3.92ER^2}{\rho}} = 65 \text{ m} \quad (10.16)$$

The above solution applies to a prismatic column of a constant cross-section.

Approximate solution can be derived from the Trefftz condition $\delta^2\Pi = 0$. Starting from Eq. (10.3) and performing the second variation one gets

$$EI \int_0^l \delta w'' \delta w' dx + \int_0^l q(l-x) \delta w' \delta w dx \quad (10.17)$$

The critical compressive body force is then

$$q = EI \frac{\int_0^l \phi'' \phi'' dx}{\int_0^l (l-x) \phi' \phi' dx} \quad (10.18)$$

As compared with the standard Trefftz formula for tip loaded column, there is the term $(l-x)$ in the denominator. As an example consider the simplest parabolic deflection shape

$$\phi = x^2 \quad (10.19a)$$

$$\phi' = 2x \quad (10.19b)$$

$$\phi'' = 2 \quad (10.19c)$$

Introducing the above expression into Eq. (10.18), the critical buckling weight per unit length is

$$q = \frac{12EI}{l^3} \quad (10.20)$$

The error in this approximation is $\frac{12 - 7.837}{7.837} = 53\%$ which is not good. As a second trial consider a power shape function with a fractional exponent α

$$\phi = x^\alpha \quad (10.21a)$$

$$\phi' = \alpha x^{\alpha-1} \quad (10.21b)$$

$$\phi'' = \alpha(\alpha - 1)x^{\alpha-2} \quad (10.21c)$$

The resulting solution is

$$q = \frac{2EI}{l^3} \frac{\alpha(\alpha - 1)(2\alpha - 1)}{2\alpha - 3} \quad (10.22)$$

The critical buckling parameter attains a minimum at $\alpha = 1.75$. The minimum buckling load is

$$q_{\min} = 9.8 \frac{EI}{l^3} \quad (10.23)$$

The error is slashed by half but it is still large at 25%. In the third attempt, let's consider the trigonometric function

$$\phi = 1 - \cos \frac{\pi x}{2l} \quad (10.24a)$$

$$\phi' = \left(\frac{\pi}{2l}\right) \sin \frac{\pi x}{2l} \quad (10.24b)$$

$$\phi'' = \left(\frac{\pi}{2l}\right)^2 \cos \frac{\pi x}{2l} \quad (10.24c)$$

In addition to satisfying clamped kinematic condition at $x = 0$, the cosine shape gives the zero bending moment at the top. Substituting Eqs. (10.24) into the Trefftz condition, Eq. (10.18), the following closed-form solution is obtained

$$q = \frac{EI}{l^3} \frac{\pi^4}{2(\pi^2 - 4)} = 8.29 \frac{EI}{l^3} \quad (10.25)$$

which differs by only 6% from the exact solution. The true shape of the column which buckles by its own weight is the Bessel function but the trigonometric function provides a very good approximation.

For over 200 years the Euler solution of buckling of a column under its own weight remains unchallenged. In 1960 Keller and Niordson asked the question by how much can the height of the column be increased. If the same volume of material is distributed as a constant cross-section prismatic structure of the radius $r = 0.1$ m, the length of the column would be

$$l = \frac{V}{\pi r^2} = \frac{1}{\pi 0.1^2} = 32 \text{ m}$$

and the weight per unit length of a still column will be

$$q = \frac{V}{l} = \frac{7.8 \times 10^4}{32} = 24 \text{ N/m}$$

Using Eq. (10.12) we can check if such a column will stay or buckle under its own weight

$$l_c^2 = 7.84 \frac{EI}{N}$$

where $I = \frac{\pi r^4}{4}$, $N = V\rho$ and $E = 2.1 \times 10^{11}$ N/m². Substituting the above values, the critical length becomes $l_c = 26$ m. This means that the 32 m prismatic column will buckle and cannot be erected. By shaping the column according to Fig. (10.2) its length can be increased by a factor of $86/26 = 3.3$.

If the cross-section is variable, this question has led to a very complex mathematical problem. Some aspects of this solution are still studied up to now. The problem is well-posed if the optimal solution is sought under a constant, given volume of the material. There is no simple closed-form solution to the problem so the answer is obtained through numerical optimization, see Fig. (10.2).

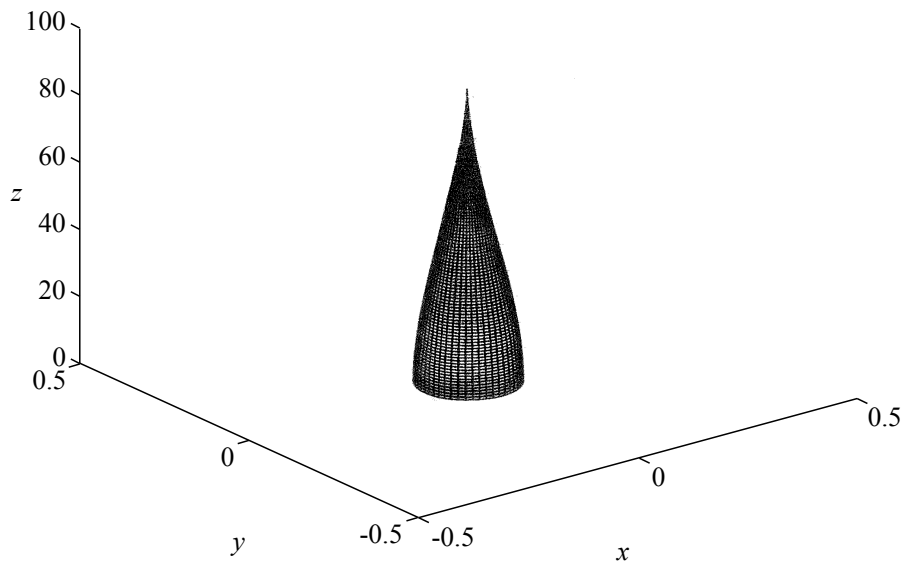


Figure 10.2: The shape of the tallest column.

Note that the height of the column was scaled down to fit on the page. To give you an idea, the steel column of the total volume of 1.0 m^3 and the bare radius of 10 cm could be as high as $l = 86$ m.

10.2 Deflection Behavior for Beam with Compressive Axial Loads and Transverse Loads

Consider a simply supported beam with a fixed load f applied at the middle as shown in Fig. (10.3). Additionally, the beam is subjected to a compressive axial load P . The total

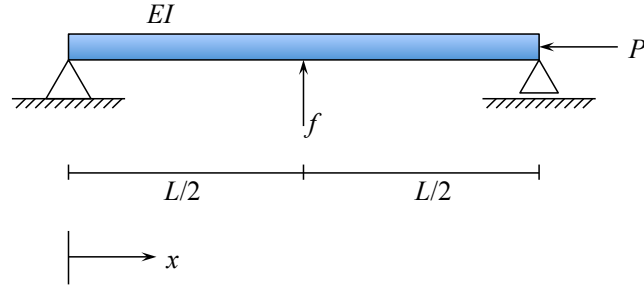


Figure 10.3: Simply supported beam with intermediate transverse load.

potential energy for this mechanical system is

$$\Pi_{\text{total}} = \int_0^L \frac{1}{2} EI (v'')^2 dx - P \int_0^L \frac{1}{2} (v')^2 dx - f v \left(\frac{L}{2} \right) \quad (10.26)$$

If $f = 0$, we are looking at a classical buckling problem; viz., the beam remains straight until a critical load is reached after which the beam bends suddenly. The critical load for the configuration shown is $P_{\text{cr}} = \pi^2 EI / L^2$. Let's investigate the behavior for $f \neq 0$.

The stationary points of the potential energy still give the solutions $v(x)$ which satisfy equilibrium. Let's compute an approximate solution using the form

$$v(x) \approx C \sin \left(\pi \frac{x}{L} \right) \quad (10.27)$$

The derivatives of this function are

$$\begin{aligned} v'(x) &= C \frac{\pi}{L} \cos \left(\pi \frac{x}{L} \right) \\ v''(x) &= -C \left(\frac{\pi}{L} \right)^2 \sin \left(\pi \frac{x}{L} \right) \end{aligned}$$

Inserting these into the potential energy yields

$$\begin{aligned} \Pi_{\text{total}} &= \int_0^L \frac{1}{2} EI \left(\frac{\pi}{L} \right)^4 C^2 \sin^2 \left(\pi \frac{x}{L} \right) dx \\ &\quad - P \int_0^L \frac{1}{2} \left(\frac{\pi}{L} \right)^2 C^2 \cos^2 \left(\pi \frac{x}{L} \right) dx - f C \sin \left(\pi \frac{L/2}{L} \right) \\ &= \int_0^L \frac{1}{2} EI \left(\frac{\pi}{L} \right)^4 C^2 \left[\frac{1}{2} - \frac{1}{2} \cos \left(\frac{2\pi x}{L} \right) \right] dx \\ &\quad - P \int_0^L \frac{1}{2} \left(\frac{\pi}{L} \right)^2 C^2 \left[\frac{1}{2} + \frac{1}{2} \cos \left(\frac{2\pi x}{L} \right) \right] dx - f C \\ &= \frac{1}{4} EI \left(\frac{\pi}{L} \right)^4 C^2 L - P \frac{1}{4} \left(\frac{\pi}{L} \right)^2 C^2 L - f C \end{aligned}$$

The stationary condition yields

$$\begin{aligned} 0 &= \frac{d\Pi_{\text{total}}}{dC} = \frac{1}{2} EI \left(\frac{\pi}{L} \right)^4 CL - P \frac{1}{2} \left(\frac{\pi}{L} \right)^2 CL - f \\ &= C \left[\frac{1}{2} EI \left(\frac{\pi}{L} \right)^4 L - P \frac{1}{2} \left(\frac{\pi}{L} \right)^2 L \right] - f = 0 \end{aligned} \quad (10.28)$$

and thus

$$\begin{aligned}
 C &= \frac{f}{\frac{EI\pi^4}{2L^3} - P\frac{\pi^2}{2L}} \\
 &= \frac{f2L/\pi^2}{\frac{EI\pi^2}{L^2} - P} \\
 &= \frac{2L}{\pi^2} \frac{f}{P_{cr} - P}
 \end{aligned} \tag{10.29}$$

The approximate solution has the form

$$v(x) \approx \frac{2L}{\pi^2} \frac{f}{P_{cr} - P} \sin\left(\pi \frac{x}{L}\right) \tag{10.30}$$

The central deflection $w_o = v(x = \frac{l}{2})$ is

$$w_o = \frac{fl^3}{48.7EI} \frac{1}{1 - \frac{P}{P_c}} \tag{10.31}$$

For zero axial load, Eq. (10.31) predicts a linear relation between the lateral point load and deflection w_o . The approximate coefficient $\frac{\pi^4}{2} \cong 48.7$ is very close to the exact value 48 for the pin-pin column loaded by the point force f . The linear relation holds also for any constant value of P/P_c . A much more interesting picture is obtained by fixing the lateral load and changing the axial load. Equation (10.31) can be written as

$$w_o = \frac{\eta}{1 - \frac{P}{P_c}}, \text{ where } \eta = \frac{fl^3}{48.7EI} \tag{10.32}$$

which is plotted in Fig. (10.4). Note that the positive force is in compression while the negative in tension. Application of the lateral force deflects the beam by the amount η . Then, on application of the in-plane compressive load, the beam-column behaves as an imperfect column. By reversing the sign of the in-plane load from compression into tension, the central deflection becomes smaller and vanishes with $P/P_c \rightarrow \infty$. This is fully consistent with our everyday experience that by tightening the rope/cable, its deflection is reduced.

10.3 Snap-through of a Two Bar System

This is a very interesting problem, because it summaries and even extends our knowledge.

There are three hinges so that each rod is a pin-pin column. The rods are elastic characterized by the bending rigidity EI , axial rigidity EA . The initial stress-free configuration is defined by the height \bar{w}_o , which was previously called the amplitude of initial imperfection.

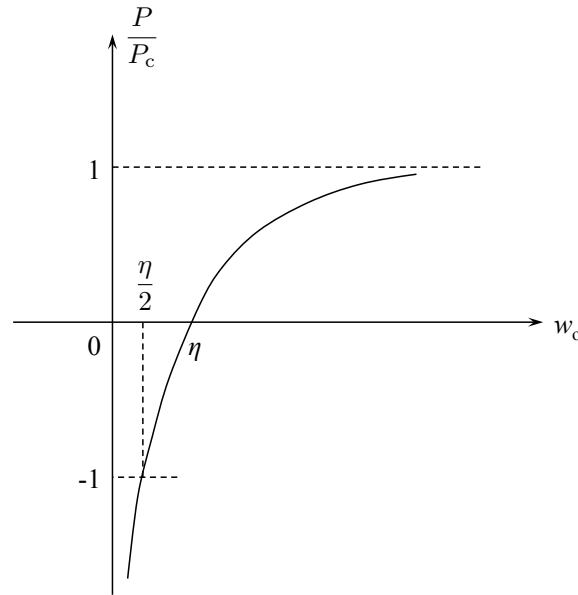


Figure 10.4: Relationship between the axial load and lateral deflections.

Here, \bar{w}_o should be regarded as the initial shape of the structure. Upon application of the load, a compressive axial force develops in the rod, their length shortens allowing for a straight (flat) configuration. The system snaps into a new configuration where tensile force develops in the rods. Depending on the slenderness ratio, they may buckle sometime during the loading process.

Pre-buckling solution

Due to the unmovable hinges, the in-plane components of the displacement is zero, $u = 0$. The strain in the bars develops by the presence of finite rotations

$$\epsilon = \frac{1}{2}(w')^2 - \frac{1}{2}(\bar{w}')^2 \quad (10.33)$$

In the pre buckling configuration the rods are straight, so

$$w' = \frac{w_o}{l}, \quad \bar{w}' = \frac{\bar{w}_o}{l} \quad (10.34)$$

and Eq. (10.33) reduces to

$$\epsilon = \frac{1}{2} \left(\frac{w_o}{l} \right)^2 - \frac{1}{2} \left(\frac{\bar{w}_o}{l} \right)^2 \quad (10.35)$$

The plot of the dimensionless strain versus the ratio w_o/\bar{w}_o is shown in Fig. (10.6).

From the elasticity law, the axial force in the rod is

$$N = EA\epsilon = \frac{EA}{2} \left[\left(\frac{w_o}{l} \right)^2 - \left(\frac{\bar{w}_o}{l} \right)^2 \right] = \begin{cases} \text{compressive for } -w_o \leq w_o \leq \bar{w}_o \\ \text{tensile for } w_o < -\bar{w}_o \end{cases} \quad (10.36)$$

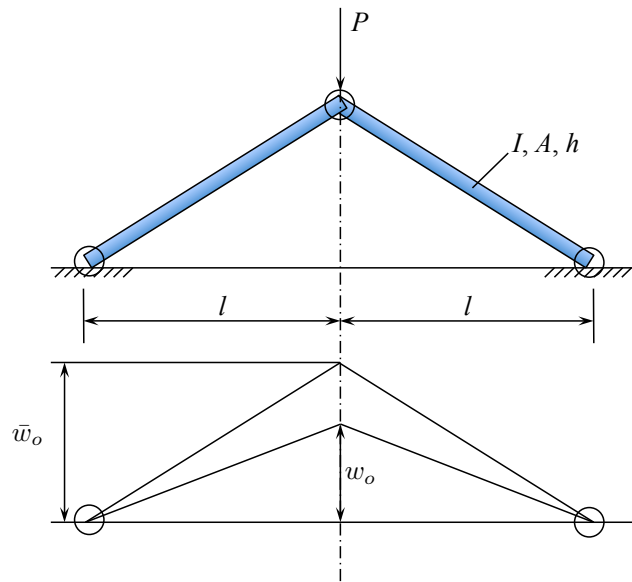


Figure 10.5: Initial and current shape of the two bar system.

Equilibrium between the external load P and the membrane force N requires that

$$P = -2N \frac{w_o}{l} \quad (10.37)$$

Eliminating the force N between Eqs. (10.36) and (10.37) yields

$$-\frac{P}{2} \frac{l}{w_o} = EA \left[\frac{1}{2} \left(\frac{w_o}{l} \right)^2 - \frac{1}{2} \left(\frac{\bar{w}_o}{l} \right)^2 \right] \quad (10.38)$$

or in a dimensionless form

$$\bar{P} = \delta(\bar{\delta}^2 - \delta^2) \quad (10.39)$$

where

$$\bar{P} = \frac{P}{EA}, \quad \delta = \frac{w_o}{l}, \quad \bar{\delta} = \frac{\bar{w}_o}{l} \quad (10.40)$$

The equilibrium path given by Eq. (10.39) is the third order parabola with three roots at $\delta = 0$, $\delta = \pm\bar{\delta}$, see Fig. (10.7).

The loading process starts at A and the portion of the trajectory AB is stable. The point B is the instability point. In the process is force controlled, there is a jump to the next equilibrium configuration which is point E. So the system “snaps” into a tensile configuration and this transition is in reality a dynamic problem. The process can be displacement control and then the force \bar{P} is the reaction force which is positive on the segments ABC and EF but negative on the segment CDE of the trajectory. This means that an opposite force \bar{P} is required on CDE to keep the system in static equilibrium. By contrast, in the force controlled process the inertia force is equilibrating the system at any time. The maximum force occurs when

$$\frac{dP}{d\delta} = \bar{\delta}^2 - 3\delta^2 = 0 \quad (10.41)$$

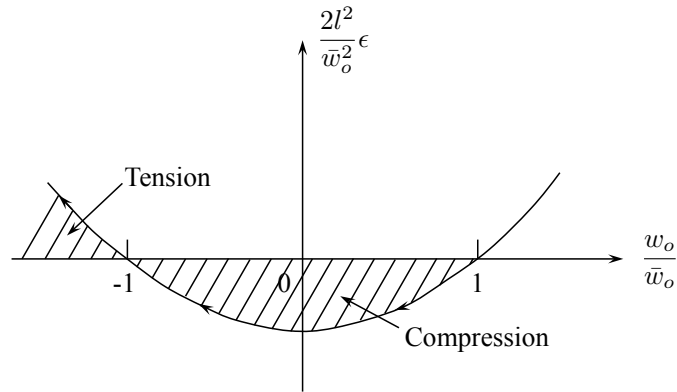


Figure 10.6: Transition from compressive into tensile strain.

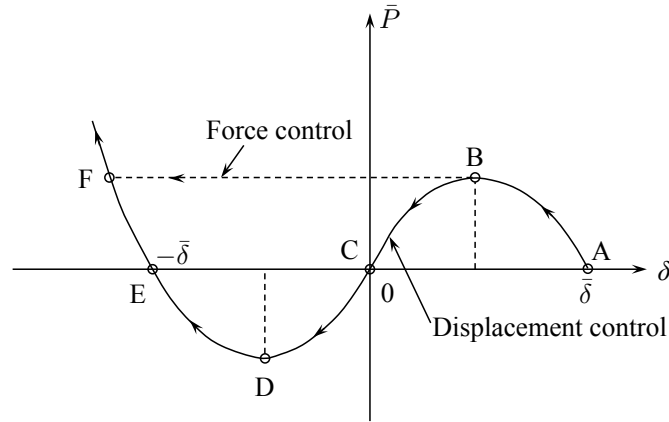


Figure 10.7: Equilibrium path in the snap-through problem.

The maximum occurs at $\delta = \bar{\delta}/\sqrt{3}$ and the maximum force is $\bar{P}_{\max} = \frac{2}{3\sqrt{3}}\bar{\delta}^3$.

Any time during the loading phase AB there is a possibility for the rods to buckle. The instant of buckling is detected by equating the axial force from Eq. (10.37) to the critical buckling force of the pin-pin column

$$N = \frac{P_{\text{cr}}}{2} \frac{l}{w_o} = \frac{\pi^2 EI}{l^2} \quad (10.42)$$

The dimensionless version of this equation is

$$\bar{P}_{\text{cr}} = \frac{P_{\text{cr}}}{EA} = 2\pi^2 \frac{\delta}{\beta^2} \quad (10.43)$$

where $\beta = \frac{l}{r}$ is the slenderness ratio, and $r^2 = \frac{I}{A}$ is the radius of gyration of the cross-section. In the coordinate system (\bar{P}, δ) , the buckling point is determined by the intersection

of the straight line, Eq. (10.42), and the third order parabola

$$\frac{2\pi^2}{\beta^2}\delta = \delta(\bar{\delta}^2 - \delta_c^2) \quad (10.44)$$

The displacement to buckle is

$$\delta_c = \sqrt{\bar{\delta}^2 - \frac{2\pi^2}{\beta^2}} \quad (10.45)$$

and the corresponding buckling force \bar{P}_c is

$$\bar{P}_c = \frac{2\pi^2}{\beta^2} \sqrt{\bar{\delta}^2 - \frac{2\pi^2}{\beta^2}} \quad (10.46)$$

The graphical interpretation of the above analysis is shown in Fig. (10.8).

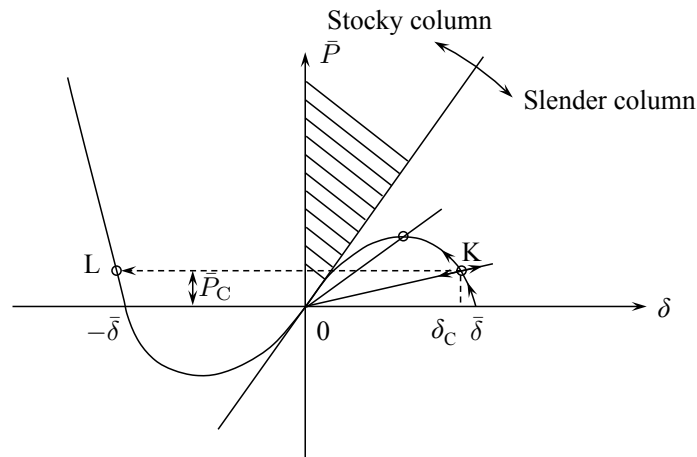


Figure 10.8: Non-linear pre-buckling path intersects with a linear post-buckling path.

There is a family of straight lines with the slenderness ratio as a parameter. The critical slenderness ratio for which buckling will never occur is

$$\beta_{\text{cr}}^2 = \frac{2\pi}{\bar{\delta}} \quad (10.47)$$

This situation corresponds to the straight line tangent to the third order parabola. Of practical interest is the situation in which the bifurcation point occurs before the maximum force is reached at $\delta_{\text{max}} = \bar{\delta}/\sqrt{3}$ and $\bar{P}_{\text{max}} = \frac{2}{3\sqrt{3}}\bar{\delta}^3$. The corresponding minimum slenderness ratio, calculated from Eq. (10.42) is

$$\beta_{\text{min}}^2 = \frac{\sqrt{3}\pi}{\bar{\delta}} \quad (10.48)$$

To sum up, there are three ranges of the slenderness ratio:

Table 10.1: Ranges of buckling response

	$\beta_{\min} < \beta < \infty$	$\beta = \beta_{\min}$	$\beta = \beta_{\min}$
\bar{P}_{\max}	$\frac{2\pi^2}{\beta^2} \sqrt{\bar{\delta}^2 - \frac{2\pi^2}{\beta^2}}$	$\frac{2}{3\sqrt{3}} \bar{\delta}^3$	No buckling static or dynamic equilibrium path
δ_{\max}	$\sqrt{\bar{\delta}^2 - \frac{2\pi^2}{\beta^2}}$	$\frac{\bar{\delta}}{\sqrt{3}}$	

For the square cross-section $h \times h$, the critical combination of the geometrical parameters

$$\frac{\bar{w}_o}{h} = 36\pi \frac{h}{l} \quad (10.49)$$

From the above solution, we conclude that snap-through of the bar system without buckling will occur only for very shallow systems.

10.4 Dynamic Snap-Through

The present lecture notes are restricted to static and quasi-static problems. However, the nature of the snap-through problem calls for the consideration of the full dynamic analysis. Assume that the loading of the two-bar system is load controlled. There is a stable equilibrium path on the portion AB. When $\bar{P}_{\max} = \frac{2}{3\sqrt{3}} \bar{\delta}^3$ is reached, the system jumps instantaneously to the next equilibrium point F in the static solution. The magnitude of the force is the same, but the corresponding displacement is determined from the solution of the cubic equation

$$\frac{2}{3\sqrt{3}} \bar{\delta}^3 = \delta \bar{\delta}^2 - \delta^3 \quad (10.50)$$

This equation has three real roots

$$\delta_1 = \frac{\bar{\delta}}{\sqrt{3}}, \delta_2 = \delta_3 = -\frac{2\bar{\delta}}{\sqrt{3}} \quad (10.51)$$

By adding inertia forces into the equation of equilibrium, the snap-through process occurs in time. The bar system is first accelerated on the portion BCD of the descending force and then decelerated on the rising portion DEF.

The dynamic solution is straight forward if the distributed mass of the rod is lumped into two discrete point masses $m = lA\rho$, as shown in Fig. (10.9).

By adding d'Alambert inertia forces into static equilibrium, Eq. (10.37), one gets

$$-P - 2m \frac{\ddot{w}_o}{2} = 2N \frac{w_o}{l} \quad (10.52)$$

where now P is positive in tension.

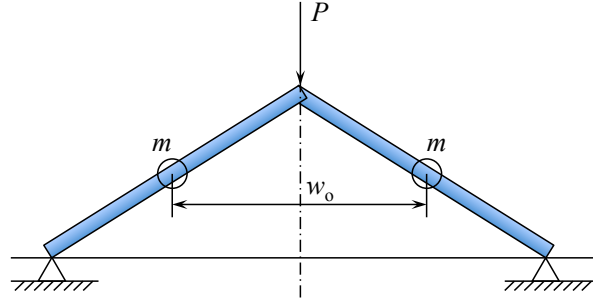


Figure 10.9: The equivalent two massless bars and two lumped masses.

Eliminating the axial force N in the bars between Eqs. (10.36) and (10.52), one gets the following differential equation

$$-\bar{P} - \frac{l^2 \rho}{E} \ddot{\delta} = \delta [\delta^2 - \bar{\delta}^2] \quad (10.53)$$

where the dot denotes differentiation with respect to time. It is convenient to introduce the dimensionless time $\bar{t} = \frac{t}{t_1}$, where $t_1 = \frac{l}{c}$ is the reference time, and $c^2 = \frac{E}{\rho}$ is the speed of the longitudinal stress wave in a bar. In the force controlled system, the exciting term is constant $\bar{P} = \frac{2}{3\sqrt{3}} \bar{\delta}^3$. In the new dimensionless coordinate, Eq. (10.53) takes the form

$$-\bar{P} - \ddot{\delta} = \delta^3 - \bar{\delta}^2 \delta \quad (10.54)$$

where the dot denotes differentiation with respect to the dimensionless time \bar{t} . Using the chain rule of differentiation,

$$\ddot{\delta} = \frac{d\dot{\delta}}{d\bar{t}} = \frac{d\dot{\delta}}{d\delta} \frac{d\delta}{d\bar{t}} = \frac{d\dot{\delta}}{d\delta} \dot{\delta} \quad (10.55)$$

one can get a solution on the phase plane $(\delta, \dot{\delta})$ rather than in the time domain. Substituting Eq. (10.55) into Eq. (10.54), the following equation is obtained

$$-\bar{P} d\delta - \dot{\delta} d\dot{\delta} = (\delta^3 - \bar{\delta}^2 \delta) d\delta \quad (10.56)$$

which can be readily integrated to give

$$-\bar{P} \delta - \frac{1}{2} \dot{\delta}^2 = \frac{\delta^4}{4} - \frac{\bar{\delta}^2 \delta^2}{2} + C \quad (10.57)$$

The integration constant C is determined from the initial condition that the velocity $\dot{\delta}$ is zero when the deflection reaches $\delta = \frac{1}{\sqrt{3}}$ (point B). The solution for the velocity $\dot{\delta}$ is

$$\dot{\delta} = 2\bar{\delta}^2 \sqrt{-P \left(\frac{\delta}{\bar{\delta}}\right) + \frac{1}{2} \left(\frac{\delta}{\bar{\delta}}\right)^2 - \frac{1}{4} \left(\frac{\delta}{\bar{\delta}}\right)^4 + \frac{1}{12}} \quad (10.58)$$

In terms of the normalized velocity $\frac{\dot{\delta}}{2\bar{\delta}^2} = \bar{v}$ and the normalized deflection $\eta = \frac{\delta}{\bar{\delta}}$, Eq. (10.58) reads

$$\bar{v} = \sqrt{-\frac{2}{3\sqrt{3}}\eta + \frac{1}{2}\eta^2 - \frac{1}{4}\eta^4 + \frac{1}{12}} \quad (10.59)$$

The plot of \bar{v} versus η is shown in Fig. (10.10).

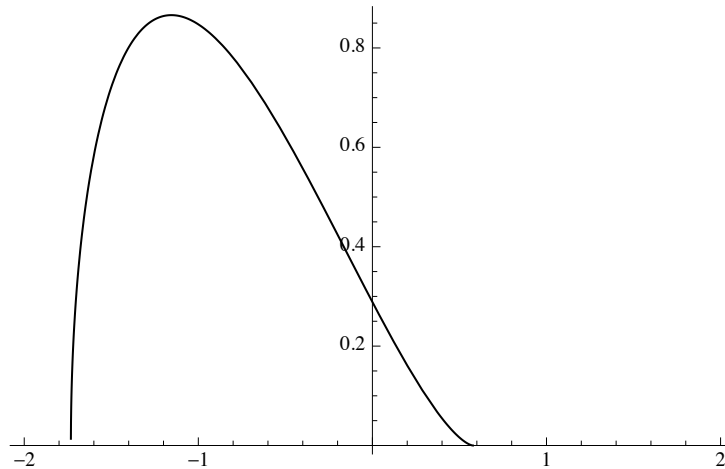


Figure 10.10: The plot of \bar{v} versus η in dynamic snap through.

The polynomial in η under the square root in Eq. (10.59) has two real roots, at $\eta = \frac{1}{\sqrt{3}}$ and $\eta = -\sqrt{3}$. The dynamic motion starts at B, increases slowly, reaches a maximum in F and falls rapidly to zero at the point G with the coordinate $\eta_f = \sqrt{3}$. Note that the dynamic deflection overshoots considerably the deflection reached in the static problem $\eta_{\text{stat}} = \frac{2}{\sqrt{3}} = 1.15$.

At the final stage when the motion of the system stops, there is enough tensile energy stored in the bar to initial free vibration with the forcing term \bar{P} removed. The solution to this phase is given by Eq. (10.57) with $\bar{P} = 0$, and the new integration constant $C_1 = \frac{3}{4}$ so that continuity of velocity is achieved. The plot of the free vibration of the system, governed by

$$\bar{v} = \sqrt{\frac{1}{2}\eta^2 - \frac{1}{4}\eta^4 + \frac{3}{4}} \quad (10.60)$$

is shown in Fig. (10.11), in comparison with the dynamic snap-through plot.

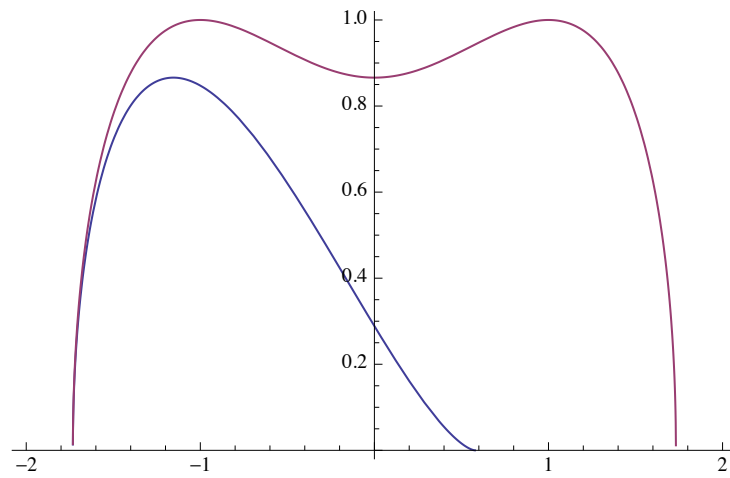


Figure 10.11: The plot of \bar{v} versus η in free vibration, in comparison with the dynamic-through plot.

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