Computational Ocean Acoustics

- Ray Tracing
- Wavenumber Integration
- Normal Modes
- Parabolic Equation

Parabolic Equation

- Mathematical Derivation (6.2)
  - Phase Errors and Angular Limitations (6.2.4)
- Starting Fields (6.4)
  - Modal starter
  - PE Self Starter
  - Analytical Starters
- PE Solvers
  - Split-Step Fourier Algorithm (6.5)
  - PE Solutions using FD and FEM (6.6)

Split-Step PEs

Square-root operator, Feit–Fleck splitting

$$
\begin{aligned}
Q &= \sqrt{1+\varepsilon+\mu} \\
&\simeq \sqrt{1+\mu} + \sqrt{1+\varepsilon} - 1\,,
\end{aligned}
$$

Standard PE – $\mu \simeq 0$

$$\frac{\partial \psi}{\partial r} = \frac{ik_0}{2}\left(n^2 - 1 + \frac{1}{k_0^2}\frac{\partial^2}{\partial z^2}\right)\psi\,,$$

Thomson–Chapman PE

$$\frac{\partial \psi}{\partial r} = ik_0\left(n - 2 + \sqrt{1 + \frac{1}{k_0^2}\frac{\partial^2}{\partial z^2}}\right)\psi\,.$$

LOGPE

$$\frac{\partial \psi}{\partial r} = ik_0\left\{\ln n + \frac{1}{2}\ln\left[\cos^2\left(-\frac{i}{k_0}\frac{\partial}{\partial z}\right)\right]\right\}\psi\,,$$

Padé Approximation

$$\sqrt{1+q} = 1 + \sum_{j=1}^{m} \frac{a_{j,m}\, q}{1 + b_{j,m}\, q} + O(q^{2m+1}) \,,$$

$$a_{j,m} = \frac{2}{2m+1} \sin^2\left(\frac{j\pi}{2m+1}\right) ,$$

$$b_{j,m} = \cos^2\left(\frac{j\pi}{2m+1}\right) .$$

First-order Padé Approximation

$$\sqrt{1+q} \simeq 1 + \frac{0.50\, q}{1 + 0.25\, q} = \frac{1 + 0.75q}{1 + 0.25q} \,,$$

Second-order Padé Approximation

$$\sqrt{1+q} \simeq 1 + \frac{0.13820\, q}{1 + 0.65451\, q} + \frac{0.36180\, q}{1 + 0.09549\, q} \,,$$

Very-Wide-Angle Padé Parabolic Equation (Collins)

$$\frac{\partial \psi}{\partial r} = ik_0 \left[\sum_{j=1}^{m} \frac{a_{j,m}\left(n^2 - 1 + \frac{1}{k_0^2}\frac{\partial^2}{\partial z^2}\right)}{1 + b_{j,m}\left(n^2 - 1 + \frac{1}{k_0^2}\frac{\partial^2}{\partial z^2}\right)} \right] \psi \,,$$

Phase Errors and Angular Limitations

Claerbout's wide-angle PE

$$\frac{\partial \psi}{\partial r} = ik_0 \left(\frac{1 + 0.75\, q}{1 + 0.25\, q} - 1 \right) \psi \,,$$

Range-Independent Environment

$$\left(k^2(z) + 3k_0^2 + \frac{\partial^2}{\partial z^2} \right) \frac{\partial \psi}{\partial r} = 2ik_0 \left(k^2(z) - k_0^2 + \frac{\partial^2}{\partial z^2} \right) \psi \,.$$

Separation of Variables.

$$\psi = \Phi(r)\, \Psi(z) \,,$$

$$\underbrace{\boxed{\left[\frac{d^2\Psi}{dz^2} + k^2(z)\, \Psi \right]}}_{= k_{rm}^2 \Psi} \left(\frac{d\Phi}{dr} - 2ik_0\, \Phi \right) + \underbrace{\boxed{\left[3k_0^2 \frac{d\Phi}{dr} + 2ik_0^3\, \Phi \right]}}_{= - k_{rm}^2 (\Phi' - 2ik\, \Phi)} \Psi = 0 \,,$$

Phase Errors and Angular Limitations

Vertical 'Modal' Equation

$$\frac{d^2\Psi}{dz^2} + \left[k^2(z) - k_{rm}^2\right]\Psi = 0\,,$$

PE Propagates Normal Modes Undistorted

Horizontal Parabolic Equation

$$\frac{d\Phi}{dr} - ik_0\,\frac{2k_{rm}^2 - 2k_0^2}{3k_0^2 + k_{rm}^2}\,\Phi = 0\,.$$

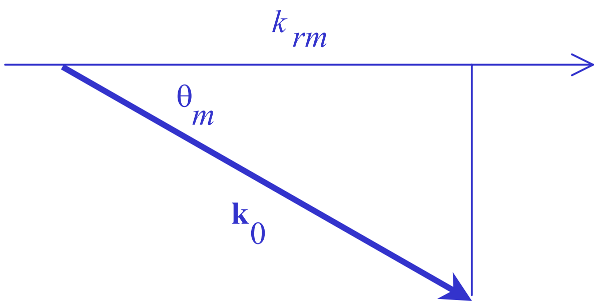
Horizontal Phase Error

Radial Solution

$$\Phi(r) = \Phi(r_0)\ \exp\left[ik_0\left(\frac{2k_{rm}^2 - 2k_0^2}{3k_0^2 + k_{rm}^2}\right)(r - r_0)\right].$$

Acoustic Pressure

$$p(r,z) = p(r_0,z)\,\sqrt{\frac{r_0}{r}}\ \exp\left[ik_0\left(\frac{k_0^2 + 3k_{rm}^2}{3k_0^2 + k_{rm}^2}\right)(r - r_0)\right].$$



Phase Errors and Angular Limitations

Exact Modal Phase

$$\exp[ik_{rm}(r - r_0)]$$

$$\begin{aligned} k_{rm} &= k_0 \cos\theta_m = k_0 \varphi \\ \varphi &= cos(\theta_m) = \sqrt{1-\sin^2\theta}\,, \end{aligned} \qquad \text{Helmholtz}$$

Clairbout Modal Phase

$$\begin{aligned} \varphi &= \frac{1+3\cos^2\theta_m}{3+\cos^2\theta_m} \\ &= \frac{1-0.75\,\sin^2\theta_m}{1-0.25\,\sin^2\theta_m}\,. \end{aligned} \qquad \text{Claerbout}$$

[See Fig 6.1 in Jensen, Kuperman, Porter and Schmidt. *Computational Ocean Acoustics.* New York: Springer-Verlag, 2000.]

PE Modal Phases

$$Q = \sqrt{1-\sin^2\theta_m}\,, \qquad \text{Helmholtz}$$

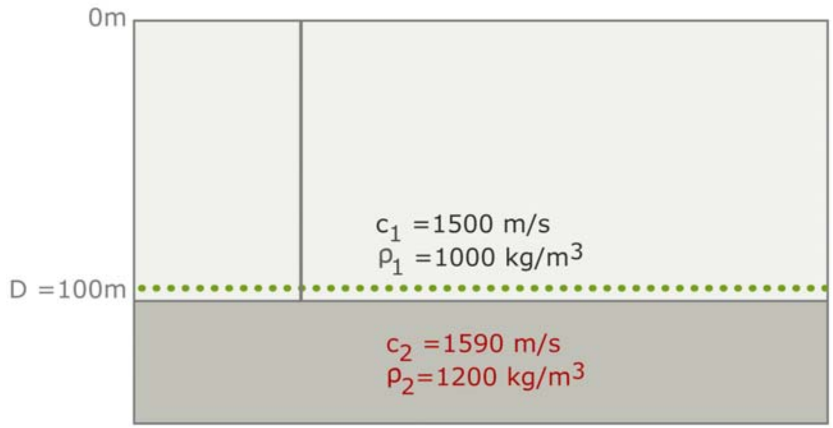
$$Q_1 = 1 - \frac{\sin^2\theta_m}{2}\,, \qquad \text{Tappert}$$

$$Q_2 = \frac{1-0.75\,\sin^2\theta_m}{1-0.25\,\sin^2\theta_m}\,, \qquad \text{Claerbout, Padé (1)}$$

$$Q_3 = \frac{0.99987-0.79624\,\sin^2\theta_m}{1-0.30102\,\sin^2\theta_m}\,, \qquad \text{Greene}$$

$$Q_4 = 1 - \frac{0.13820\,\sin^2\theta_m}{1-0.65451\,\sin^2\theta_m} - \frac{0.36180\,\sin^2\theta_m}{1-0.09549\,\sin^2\theta_m}\,. \qquad \text{Padé (2)}$$

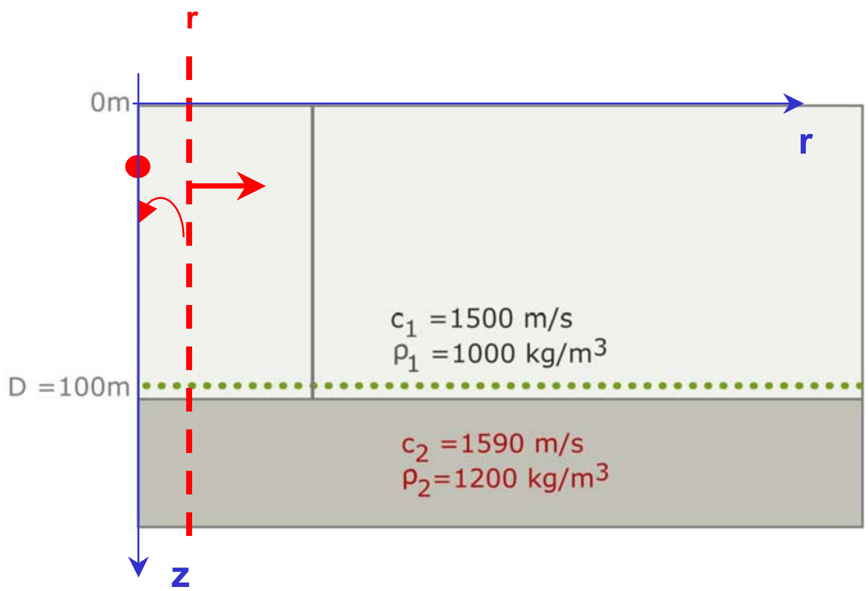
PE Workshop Case 3B



[See Jensen Fig. 6.2]

Numerical Starters

Modal Starter



$$p(r,z) = \frac{\psi(r,z)}{\sqrt{r}}\, e^{i(k_0 r - \frac{\pi}{4})}\,.$$

$$\mathrm{TL} = -20\log\frac{|\psi|}{\sqrt{r}}\,,$$

Normalized Modal Field

$$p(r,z) = \frac{1}{\rho(z_s)}\sqrt{\frac{2\pi}{r}}\sum_{m=1}^{M}\frac{\Psi_m(z_s)\,\Psi_m(z)}{\sqrt{k_{rm}}}\, e^{i(k_{rm} r - \frac{\pi}{4})}\,,$$

Normalized Starting Field

$$\psi(0,z) = \frac{\sqrt{2\pi}}{\rho(z_s)}\sum_{m=1}^{M}\frac{\Psi_m(z_s)\,\Psi_m(z)}{\sqrt{k_{rm}}}\,,$$

PE Self Starter

Helmholtz Equation in Plane Geometry

$$\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial z^2} + k^2 p = \delta(x)\,\delta(z - z_s),$$

Forward propagation

$$\frac{\partial p}{\partial x} - i\sqrt{k^2 + \frac{\partial^2}{\partial z^2}} p = 0$$

Integrate Helmholtz Equation $x = [-\epsilon, \epsilon]$

$$\lim_{x \to 0^+} \frac{\partial p}{\partial x} = \tfrac{1}{2}\delta(z - z_s).$$

$$\Rightarrow$$

$$\sqrt{k^2 + \frac{\partial^2}{\partial z^2}} p = -\frac{i}{2}\delta(z - z_s)$$

Rewrite

$$\frac{k^2 + \frac{\partial^2}{\partial z^2}}{\sqrt{k^2 + \frac{\partial^2}{\partial z^2}}} p = -\frac{i}{2}\delta(z - z_s)$$

Starting Pressure

$$\left(k^2 + \frac{\partial^2}{\partial z^2}\right) p_0 = -\frac{i}{2}\delta(z - z_s)$$

$$p_0 = \frac{1}{\sqrt{k^2 + \frac{\partial^2}{\partial z^2}}} p$$

At $x = x_0$

$$p = \left(k^2 + \frac{\partial^2}{\partial z^2}\right)^{\frac{1}{2}} p_0$$

$$= ik_0 \prod_{j=1}^{m} \frac{k_0^2 + \alpha_{j,m}(k^2 + \frac{\partial^2}{\partial z^2} - k_0^2)}{k_0^2 + \beta_{j,m}(k^2 + \frac{\partial^2}{\partial z^2} - k_0^2)} p_0$$

Cylindrical Geometry: $1/k_r m \rightarrow 1/sqrtk_r m$

$$p = \left(k^2 + \frac{\partial^2}{\partial z^2}\right)^{\frac{3}{4}} p_0$$

Analytical Starters

Gaussian Source

$$\psi(0,z) = A\,e^{-\frac{(z-z_s)^2}{W^2}} \,,$$

Parabolic Equation

$n^2(r,z) = 1$

$$\frac{\partial \psi}{\partial r} = \frac{i}{2k_0}\,\frac{\partial^2 \psi}{\partial z^2} \,,$$

Fourier Transform

$$\psi(r,z) = \int_{-\infty}^{\infty} \psi(r,k_z)\,e^{ik_z z} dk_z \,,$$

$$\psi(r,k_z) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \psi(r,z)\,e^{-ik_z z} dz \,,$$

$$\int_{-\infty}^{\infty} \left(\frac{\partial}{\partial r} - \frac{i}{2k_0}\,\frac{\partial^2}{\partial z^2}\right)\psi(r,z)\,e^{-ik_z z} dz = 0 \,.$$

$$\int_{-\infty}^{\infty} \frac{\partial^2\,\psi(r,z)}{\partial z^2}\,e^{-ik_z z} dz = -k_z^2\,\psi(r,k_z) \,,$$

Spectral Domain PE

$$\left(\frac{\partial}{\partial r} + \frac{ik_z^2}{2k_0}\right)\psi(r,k_z) = 0 \,,$$

Vertical Wavenumber Propagator

$$\psi(r,k_z) = \psi(0,k_z)\,e^{-\frac{ik_z^2 r}{2k_0}} \,.$$

Initial Condition

$$\begin{aligned}
\psi(0,k_z) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} A\,e^{-\frac{(z-z_s)^2}{W^2}}\,e^{-ik_z z} dz \\
&= \frac{A}{2\pi}\,e^{-ik_z z_s}\int_{-\infty}^{\infty} e^{-\frac{t^2}{W^2}}\,e^{-ik_z t} dt \,, \\
t &= z - z_s .
\end{aligned}$$

Analytical Starters

Vertical Wavenumber solution

Solution

$$\psi(r,k_z) = \psi(0,k_z)\, e^{-\frac{ik_z^2 r}{2k_0}} \, .$$

Initial Condition

$$\begin{aligned}
\psi(0,k_z) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} A\, e^{-\frac{(z-z_s)^2}{W^2}}\, e^{-ik_z z} dz \\
&= \frac{A}{2\pi}\, e^{-ik_z z_s} \int_{-\infty}^{\infty} e^{-\frac{t^2}{W^2}}\, e^{-ik_z t} dt \, , \\
t &= z - z_s .
\end{aligned}$$

Solution

$$\int_{-\infty}^{\infty} e^{-at^2}\, e^{\pm ibt} dt = \sqrt{\frac{\pi}{a}}\, e^{-\frac{b^2}{4a}} \, ,$$

Transformed Initial Field

$$\psi(0,k_z) = \frac{AW}{2\sqrt{\pi}}\, e^{-ik_z z_s}\, e^{-\frac{k_z^2 W^2}{4}} \, .$$

Inverse Transform

$$\psi(r,z) = \frac{AW}{2\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-\left(\frac{W^2}{4}+\frac{ir}{2k_0}\right)k_z^2}\, e^{i(z-z_s)k_z} dk_z \, .$$

Analytical Starters

Starting Field

$$\psi(r,z) = \frac{A}{\sqrt{1+\frac{i2r}{k_0 W^2}}} \exp\left[-\frac{(z-z_s)^2}{W^2\left(1+\frac{i2r}{k_0 W^2}\right)}\right].$$

$$|p|^2 = r^{-1}\psi\psi^*$$

$$\varepsilon = k_0^2 W^4 / 4r^2$$

$$|p|^2 = \frac{k_0 A^2 W^2}{2r^2\sqrt{1+\varepsilon}} \exp\left[-\frac{k_0^2 W^2 (z-z_s)^2}{2r^2(1+\varepsilon)}\right],$$

$$|p|^2 \simeq \frac{k_0 A^2 W^2}{2r^2}\left[1 - \frac{k_0^2 W^2}{2r^2}(z-z_s)^2\right].$$

Normalized Point-source Field

$$|p|^2 = \frac{1}{R^2}; \qquad R^2 = r^2 + (z-z_s)^2.$$

$$= \frac{1}{r^2\left[1+\frac{(z-z_s)^2}{r^2}\right]},$$

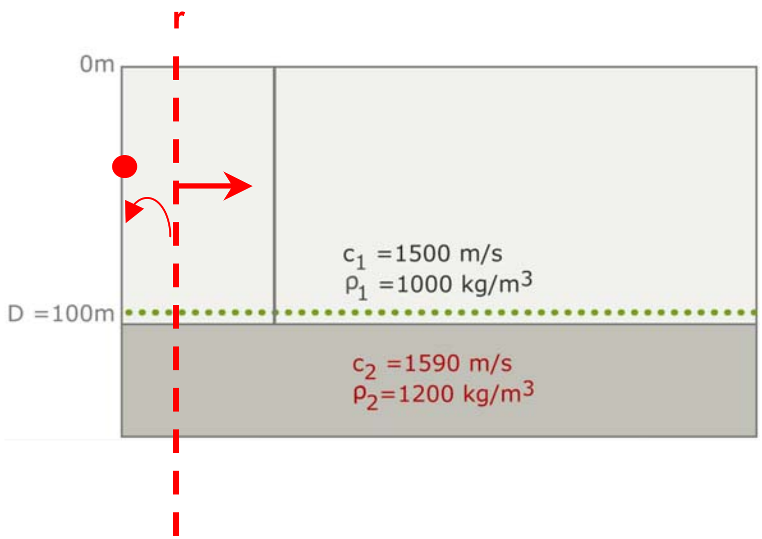
$$|p|^2 \simeq \frac{1}{r^2}\left[1 - \frac{(z-z_s)^2}{r^2}\right].$$

$$A = \sqrt{k_0}\,, \qquad W = \frac{\sqrt{2}}{k_0}.$$

Gaussian Starting Field

$$\psi(0,z) = \sqrt{k_0}\, e^{-\frac{k_0^2}{2}(z-z_s)^2},$$

Analytical Starters



Gaussian Source

$$\psi(0,z) = \sqrt{k_0}\, e^{-\frac{k_0^2}{2}(z-z_s)^2},$$

Matches point source field for $r >> z - z_s$)

Greene's Source

$$\psi(0,z) = \sqrt{k_0}\left[1.4467 - 0.4201\, k_0^2\,(z-z_s)^2\right] e^{-\frac{k_0^2(z-z_s)^2}{3.0512}},$$

Thomson's Source

$$\psi(0,k_z) = \sqrt{\frac{8\pi}{k_0}}\,\sin(k_z z_s)\left(1-\frac{k_z^2}{k_0^2}\right)^{-1/4},$$

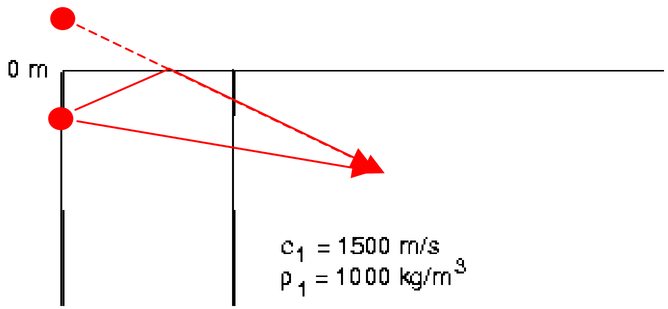
$$\psi(0,z) = F^{-1}\left\{\psi(0,k_z)\right\}$$

Generalized Gaussian Source

$$\psi(0,z) = \sqrt{k_0}\;\tan\theta_1\, e^{-\frac{k_0^2}{2}(z-z_s)^2\tan^2\theta_1}\; e^{ik_0(z-z_s)\sin\theta_2},$$

Spectral Properties of Sources

Lloyd-Mirror Halfspace Problem



Surface Condition

$$\psi(0,z) = \psi(0,z-z_s) - \psi(0,z+z_s)\,,$$

$$p(r,z) = \frac{e^{ikR_1}}{R_1} - \frac{e^{ikR_2}}{R_2}\,,$$

$$R_1 = \sqrt{r^2+(z-z_s)^2}, \quad R_2 = \sqrt{r^2+(z+z_s)^2}\,,$$

BeamDirections

$$\sin\theta_m = (2m-1)\,\frac{\lambda}{4z_s}\,,$$

Wavenumber
Integration - FFP

[See Jensen, Fig. 6.4a

Tappert PE
45 Gaussian
Source

[See Jensen, Fig. 6.4b

Spectral Properties of Sources

Lloyd-Mirror Halfspace Problem

Tappert PE | Claerbout PE | Thomson-Chapman PE

Gaussian Source

Green Source

Thomson Source

[See Jensen, Fig 6.5]