Coordinate Systems and Separation of Variables

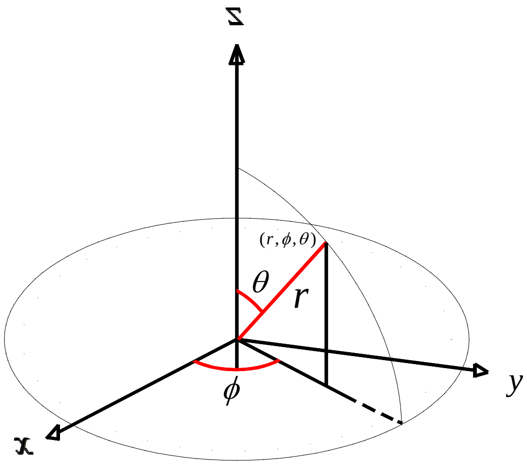
Revisiting the homogeneous wave equation… $\nabla^2\psi + \frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2} = 0$

where previously in Cartesian coordinates, the Laplacian was given by

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

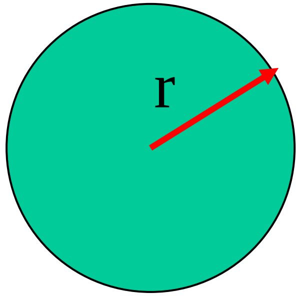
We are now faced with a spherical polar coordinate system, with the motivation that we might employ the *separation of variables* technique to solve the wave equation where spherical symmetries are involved.

Spherical Polar Coordinates



$$x = r\sin\theta\cos\phi$$
$$y = r\sin\theta\sin\phi$$
$$z = r\cos\theta$$

$$r = \sqrt{x^2 + y^2 + z^2}$$
$$\theta = \tan^{-1}\left[\sqrt{x^2 + y^2}/z\right]$$
$$\phi = \tan^{-1}\left[y/x\right]$$



$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}$$

Assuming variations only in ***r*** gives… $\nabla^2 = \frac{1}{r^2}\frac{\partial}{r}\left(r^2\frac{\partial}{\partial r}\right)$

Thus for a vibrating sphere, we can say immediately that the relevant wave equation is given by the following form of the Helmholtz equation in terms of ***r*** only…

$$\left[\frac{1}{r^2}\frac{\partial}{r}\left(r^2\frac{\partial}{\partial r}\right)+k^2\right]\psi(r)=0$$

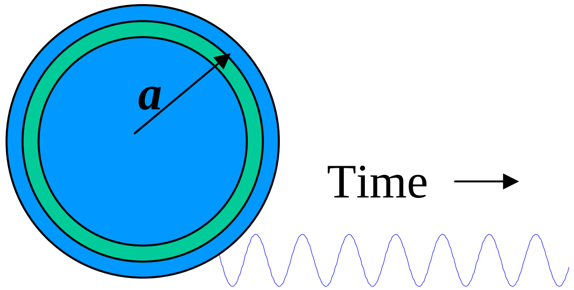
Which is known to have the two solutions…

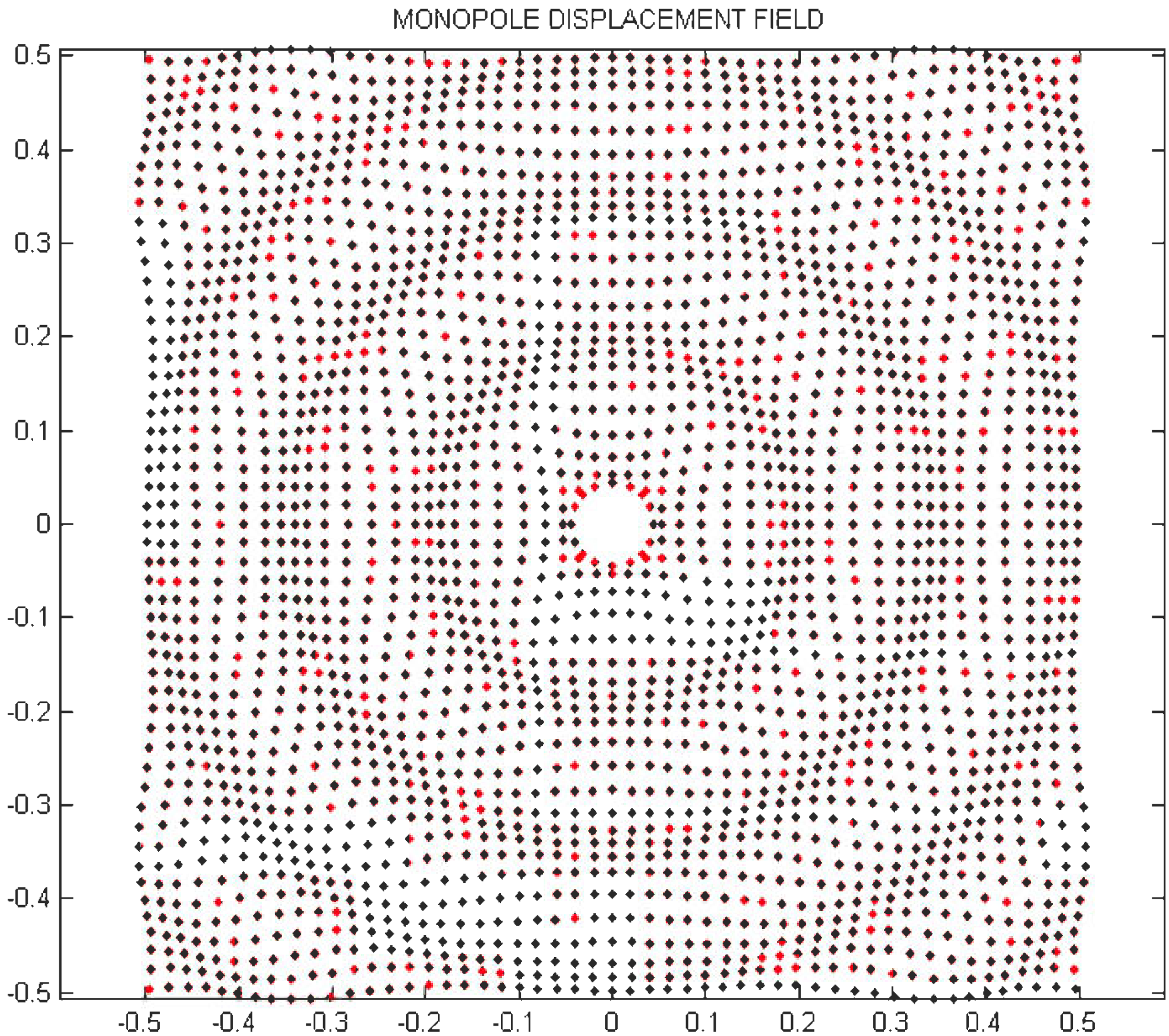
$$\psi(r)=\begin{cases}(A/r)\exp(ikr)\\(B/r)\exp(-ikr)\end{cases}$$

(Makes sense as the field decreases with ***r*** as we expect)

As we want to consider the sphere as the only source in the medium (radiation condition), we can discard the second solution, which is actually converging on the sphere, instead of propagating away from it – as must be the case.

Consider a sphere with a surface normal velocity with a sinusoidal time dependence…

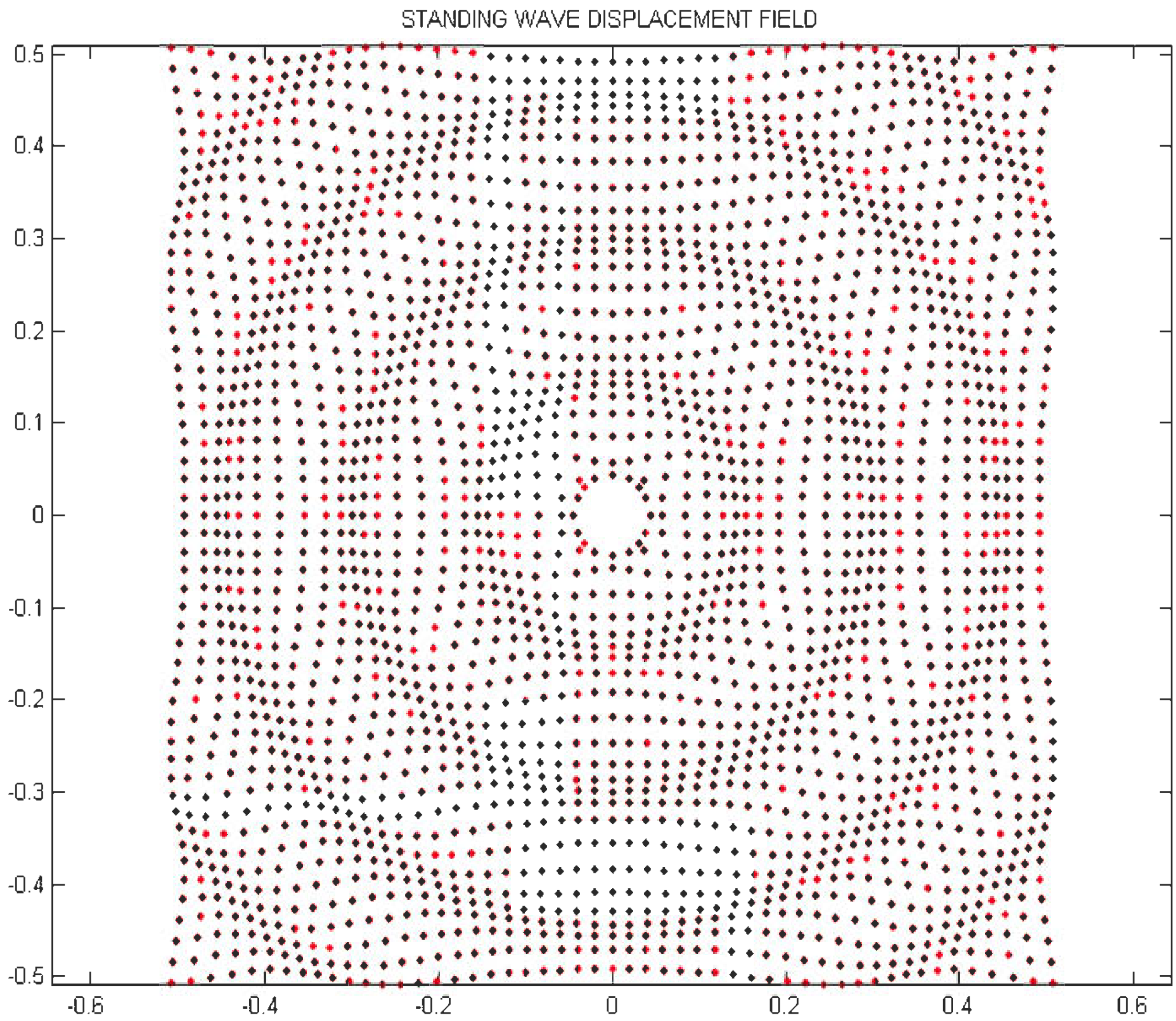




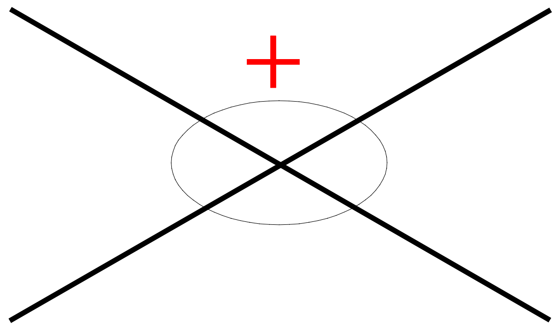
MONOPOLE DISPLACEMENT FIELD
0.5
0.4
0.3
0.2
0.1
0
-0.1
-0.2
-0.3
-0.4
-0.5
-0.5
-0.4
-0.3
-0.2
-0.1
0
0.1
0.2
0.3
0.4
0.5

Standing wave as the sum of two traveling waves

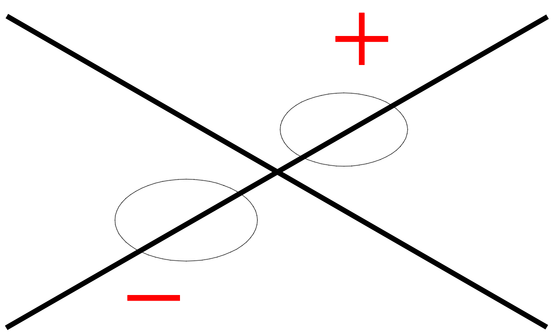
```
dt=pi/20;

for t1=0:dt:20*pi

   % Displacement field
   u=A*exp(i*k.*r0).*(i*k-1./r0);
   u=u+A*exp(i*-k.*r0).*(i*-k-1./r0);
   u=u*exp(-i*t1);
   xpos=x0+real(u).*x0./r0;
   ypos=y0+real(u).*y0./r0;

   plot(xpos,ypos,'r.')
   title('STANDING WAVE DISPLACEMENT FIELD')

   axis equal
   pause(.01);
end
```


STANDING WAVE DISPLACEMENT FIELD
0.5
0.4
0.3
0.2
0.1
0
-0.1
-0.2
-0.3
-0.4
-0.5
-0.6
-0.4
-0.2
0
0.2
0.4
0.6

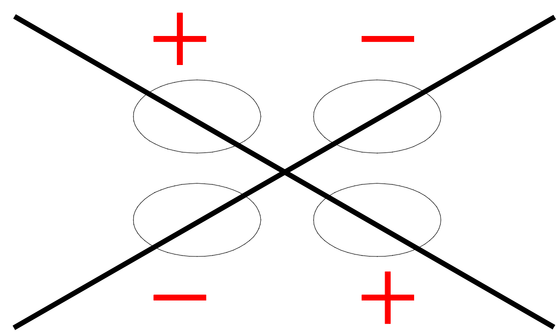
Multipoles in a plane



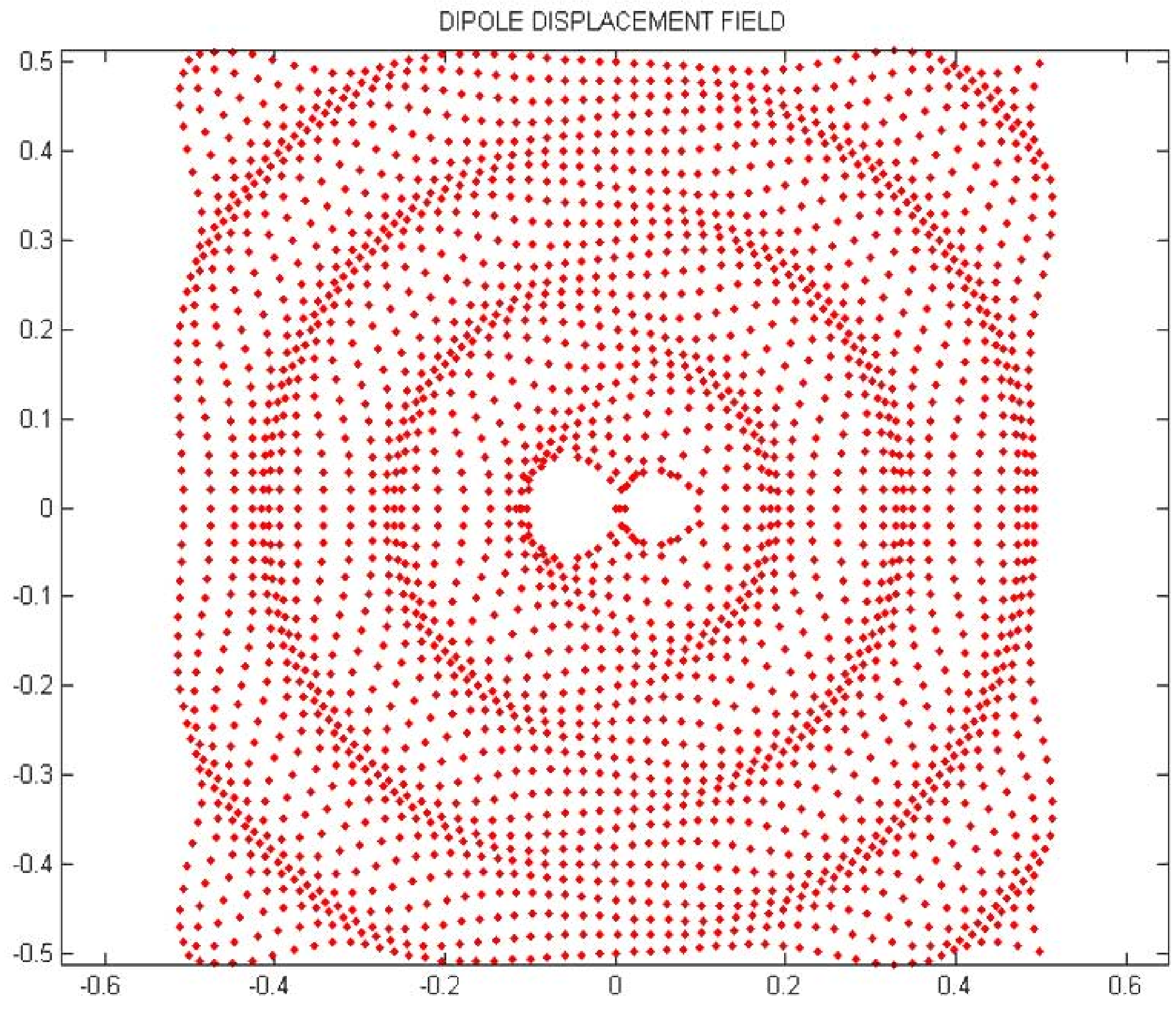
Monopole



Dipole



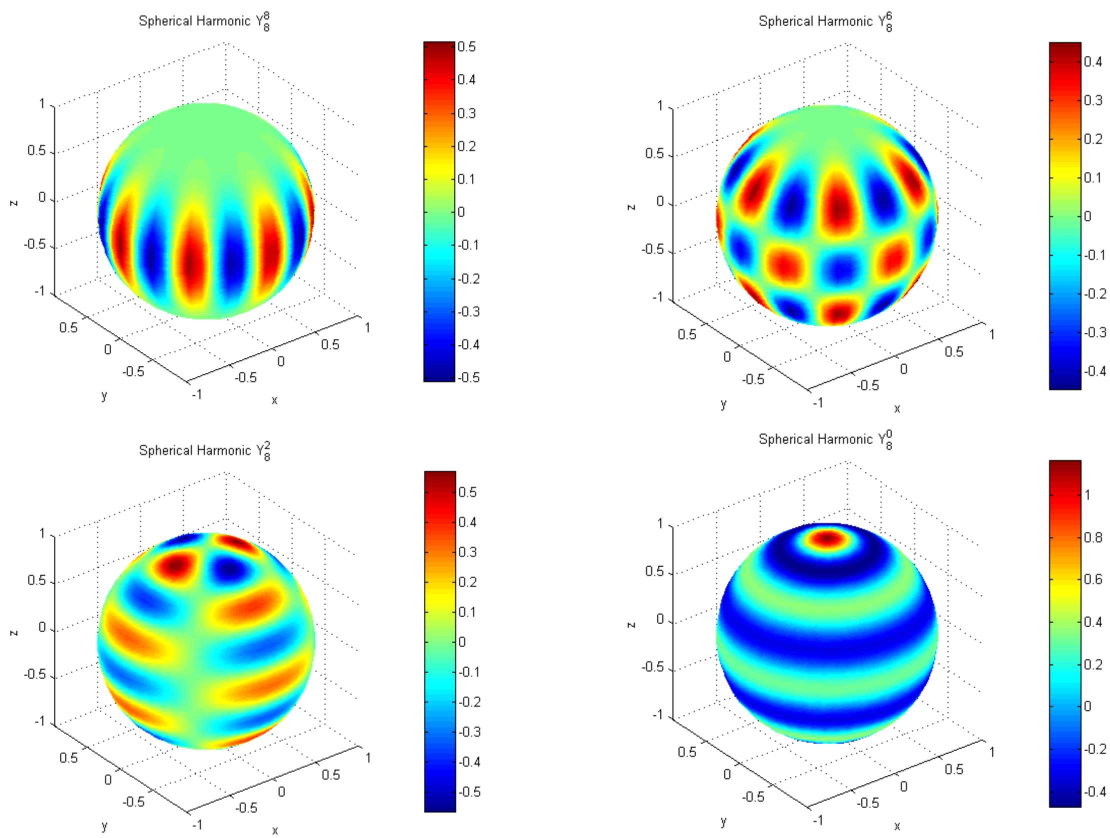
Quadrupole


DIPOLE DISPLACEMENT FIELD
0.5
0.4
0.3
0.2
0.1
0
-0.1
-0.2
-0.3
-0.4
-0.5
-0.6
-0.4
-0.2
0
0.2
0.4
0.6

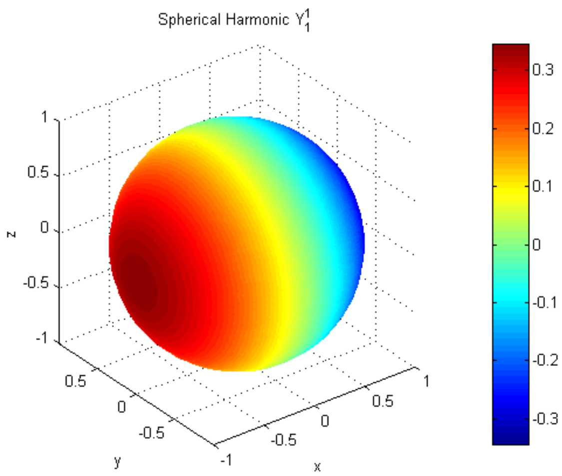
Spherical Harmonics

Lump together azmuthal and elevation dependence to arrive at spherical harmonics…

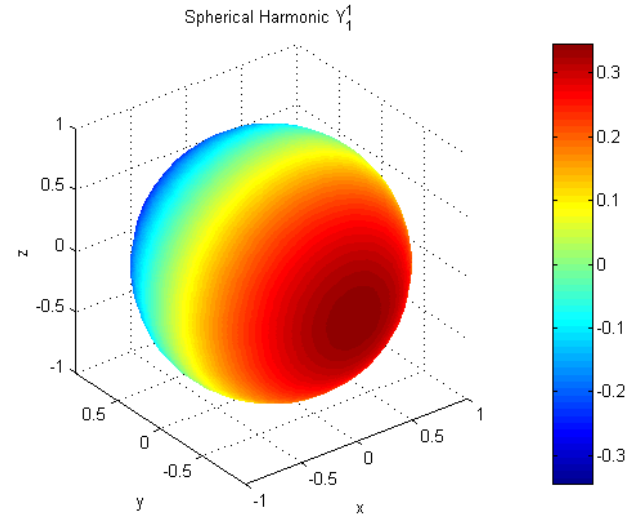
$$Y_n^m(\theta,\phi) \equiv \sqrt{\frac{(2n+1)}{4\pi}\frac{(n-m)!}{(n+m)!}}P_n^m(\cos\theta)\exp im\phi$$



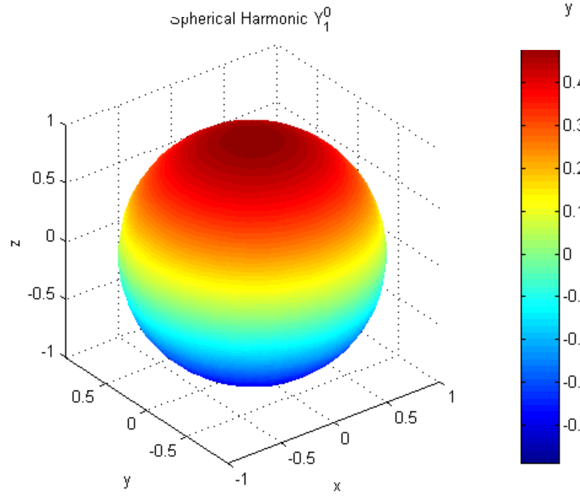
Various dipole orientations in terms of Spherical Harmonics



$\mathrm{Re}[Y_1^1]$

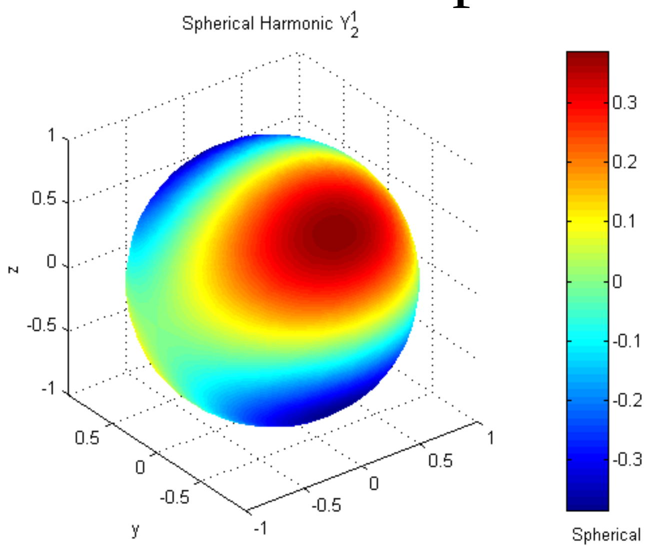


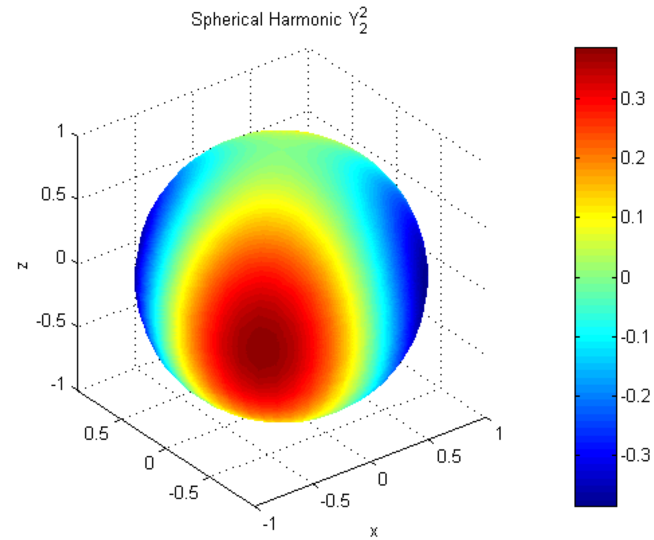
$\mathrm{Im}[Y_1^1]$



$[Y_1^0]$

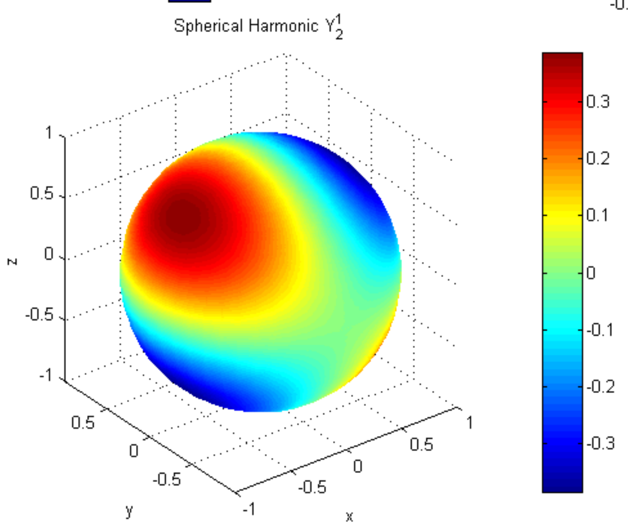
Various quadrupole orientations in terms of Spherical Harmonics





$\mathrm{Im}[Y_2^1]$

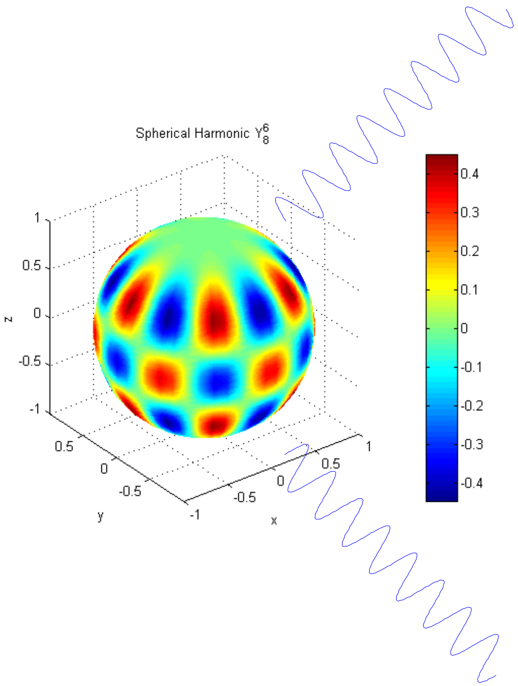
$\mathrm{Im}[Y_2^2]$



$\mathrm{Re}[Y_2^1]$

Propagating fields from spherical boundaries

$$p(r,\theta,\phi)=\sum_{n=0}^{\infty}\frac{h_n(kr)}{h_n(ka)}\sum_{m=-n}^{n}Y_n^{\,m}(\theta,\phi)\int p(a,\theta',\phi')Y_n^{\,m}(\theta',\phi')^{*}\,d\Omega'$$



Where the h_n are traveling wave solutions

$$h_n^{(1)}(x)\equiv j_n(x)+iy_n(x)=\left(\frac{\pi}{2x}\right)^{1/2}\left[J_{n+1/2}(x)+iY_{n+1/2}(x)\right]$$

$$h_n^{(2)}(x)\equiv j_n(x)-iy_n(x)=\left(\frac{\pi}{2x}\right)^{1/2}\left[J_{n+1/2}(x)-iY_{n+1/2}(x)\right]$$