Today's goal

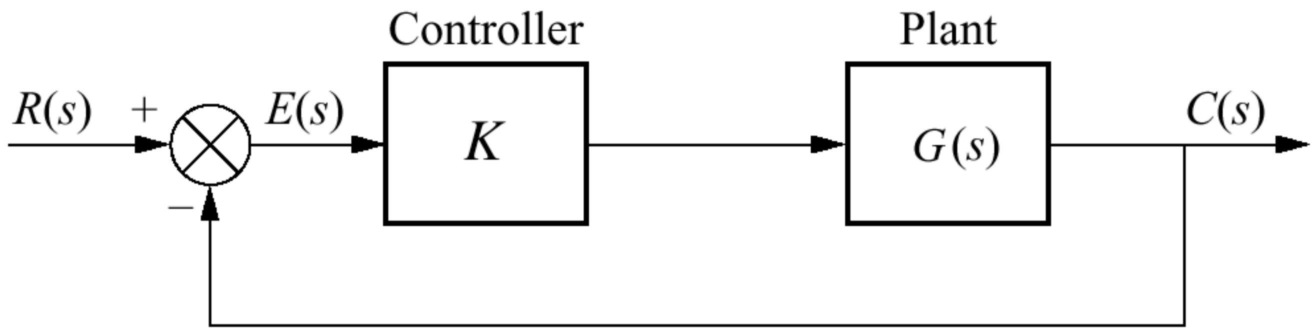
- Introduce root locus
  - Close loop transfer function &characteristic equation
  - Root locus with an example

- Rules for sketching root locus

- Observation of Root Locus with MATLAB®'s graphical user interface

Close-loop transfer function and characteristic equation

- Consider a unity feedback control loop



$$G(s) = \frac{N(s)}{D(s)}$$

- Closed-loop transfer function:

$$G_{closed}(s) = \frac{KG(s)}{1 + KG(s)} = \frac{KN(s)}{D(s) + KN(s)}$$

- The closed-loop characteristic equation:

$$1 + KG(s) = 0$$

$$\text{More generally: } D(s) + KN(s) = 0$$

What is root locus

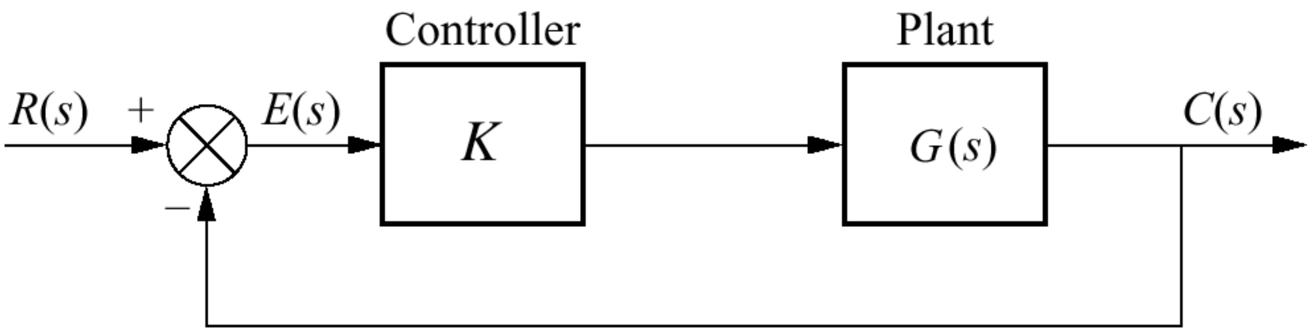
Root locus is all values of s that satisfies the system characteristic equation:

$$1 + KG(s) = 0 \;\text{ or more generally: }\; D(s) + KN(s) = 0$$

as the loop gain K varies from 0 to ∞

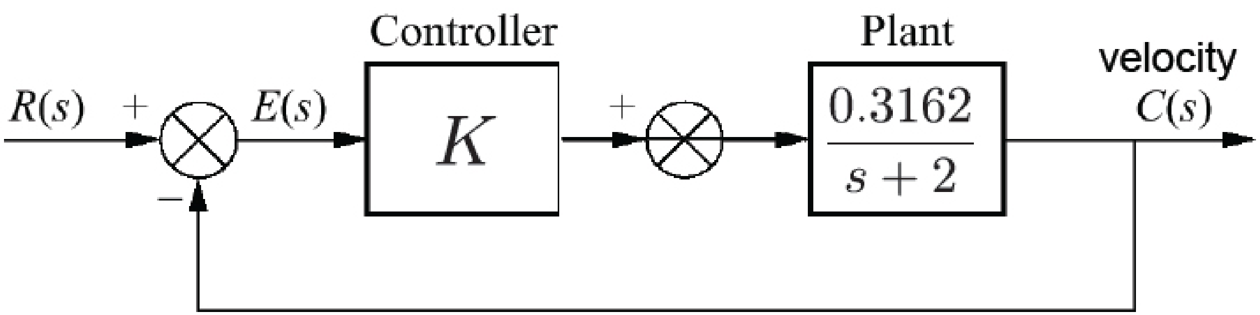
Example:

$$G(s) = \frac{0.3162}{s+2}$$



Cranking up the gain ☺

Type **0** system (no disturbance)

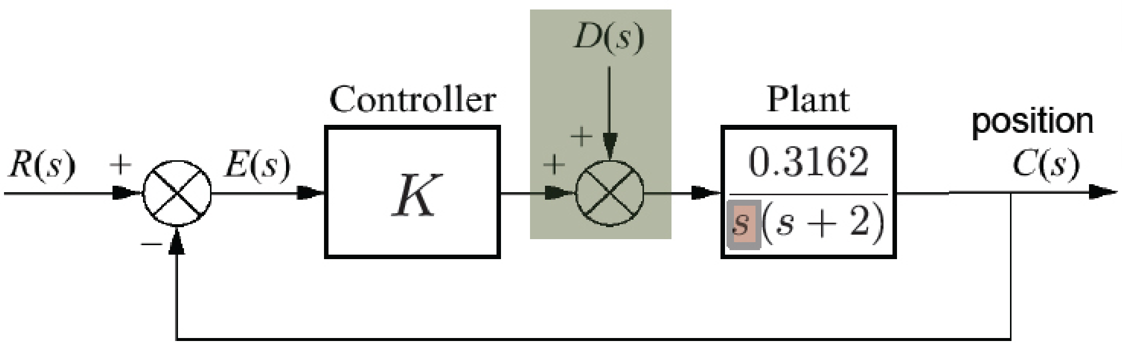


Steady–state error due to step input:

$$e_R(\infty) = \frac{2}{2 + 0.3162K}$$

$$e_R(\infty) \rightarrow 0 \quad \text{as} \quad K \rightarrow \infty$$

Type **1** system with disturbance



Steady–state error due to step input:

$$e_R(\infty) = 0$$

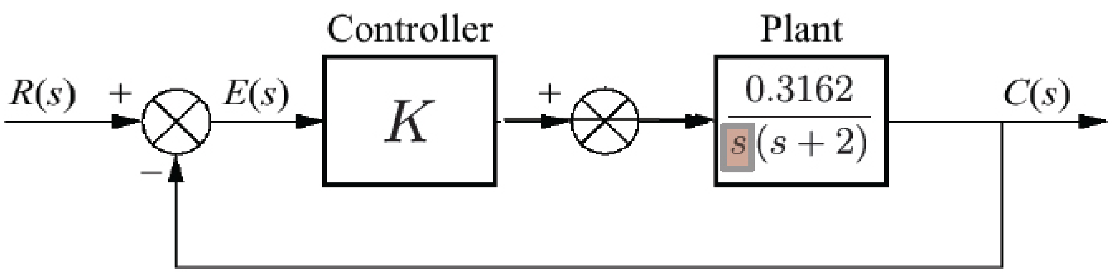
Steady–state error due to step disturbance:

$$e_D(\infty) = -\frac{1}{K}$$

$$e_D(\infty) \rightarrow 0 \quad \text{as} \quad K \rightarrow \infty$$

Cranking up the gain ☹

Type **1** system (no disturbance)



Closed–loop transfer function

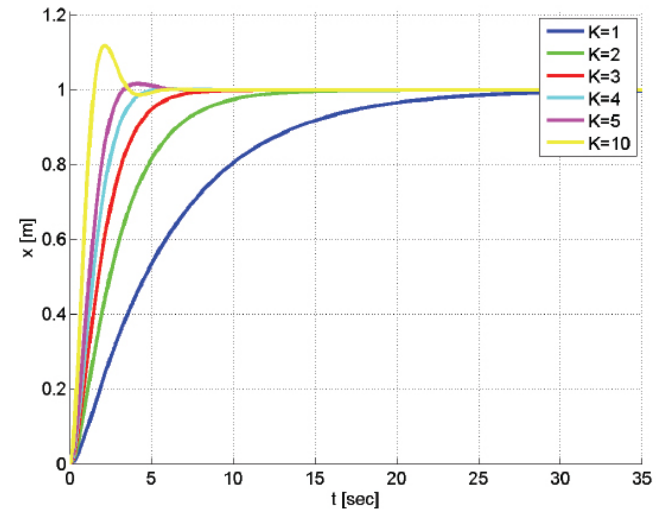
$$\frac{X(s)}{V_{ref}(s)} = \frac{0.3162K}{s^2 + 2s + 0.3162K}$$

Pole locations

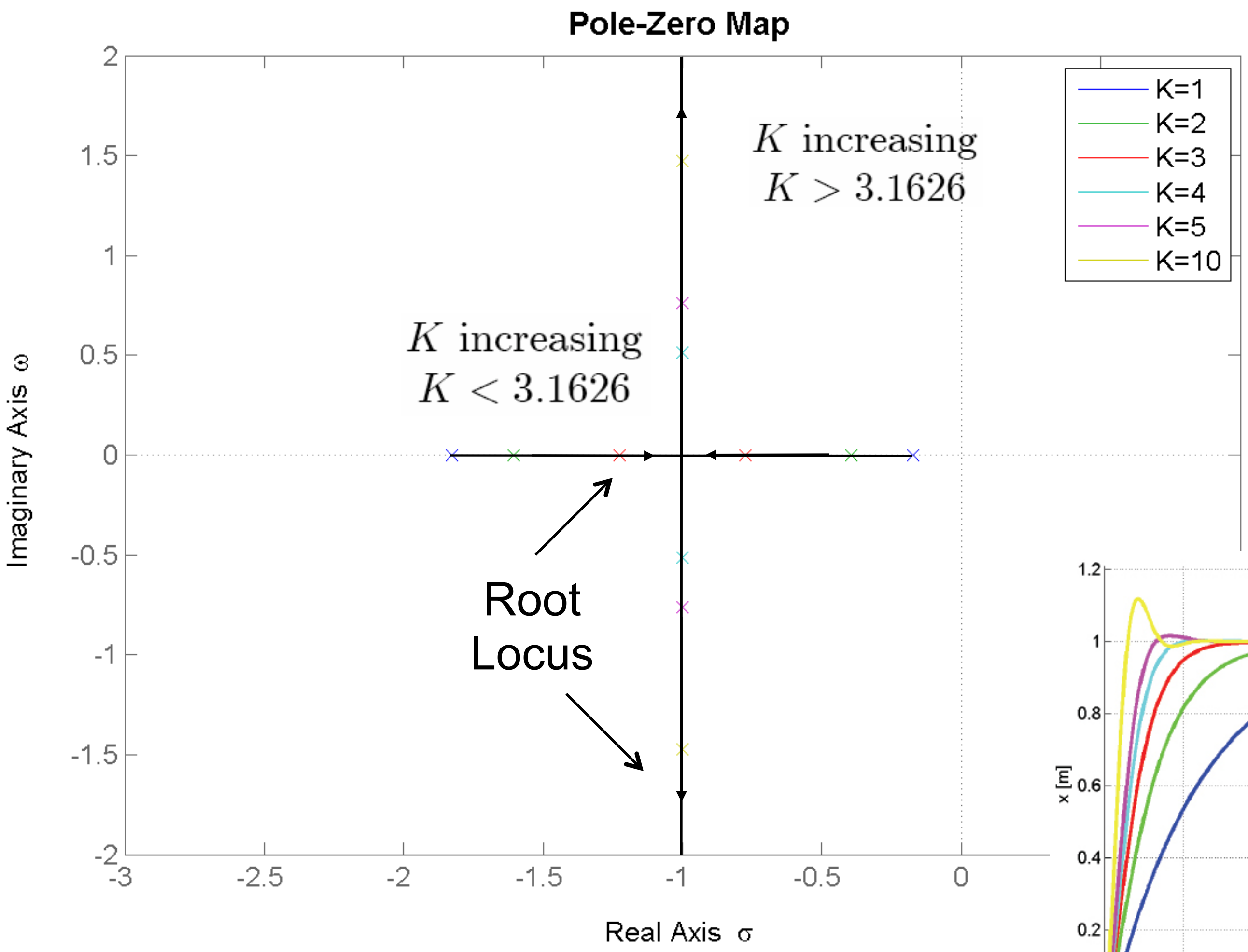
$$p_1 = -1 + \sqrt{1 - 0.3162K} \qquad p_2 = -1 - \sqrt{1 - 0.3162K}$$

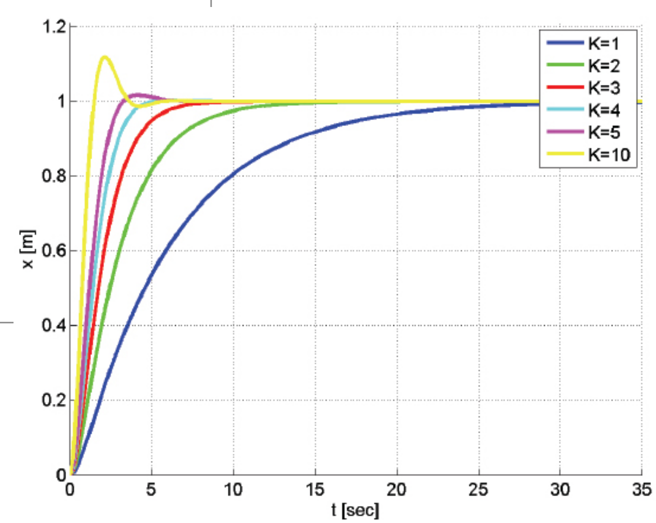
System becomes **underdamped**⇒
⇒ step response **overshoots** if

$$1 - 0.3162K < 0 \Leftrightarrow K > 3.1626$$



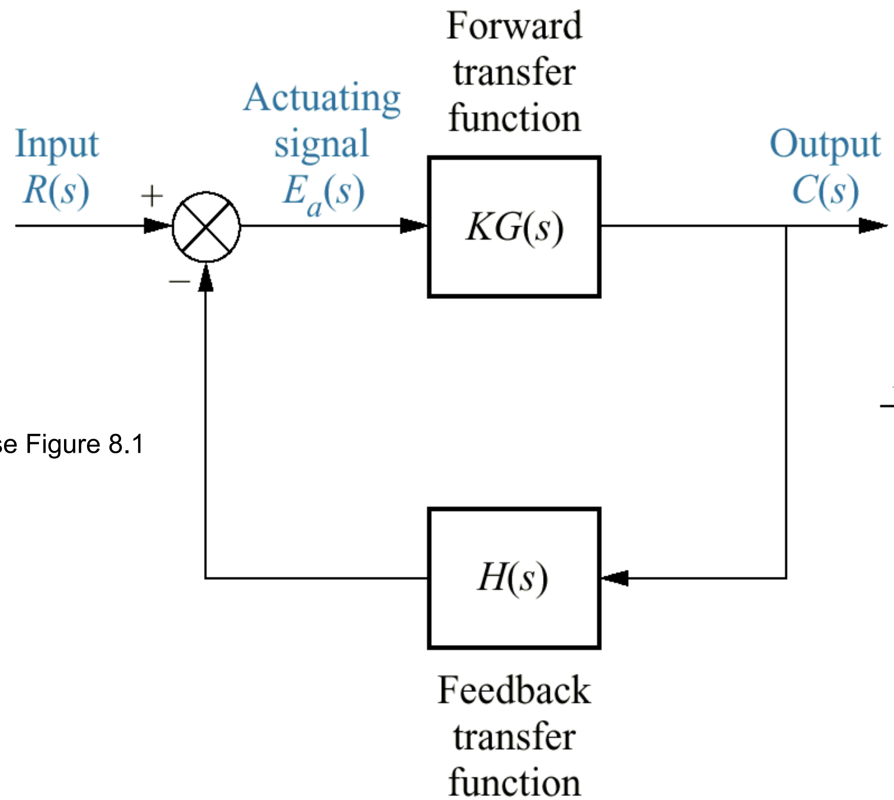
Cranking up the gain: poles and step response





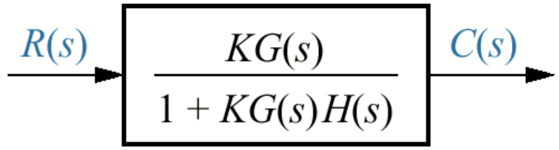
Root Locus for non-unity negative feedback systems

Caveat: K>0 (i.e., negative feedback)



Nise Figure 8.1

Closed loop TF:

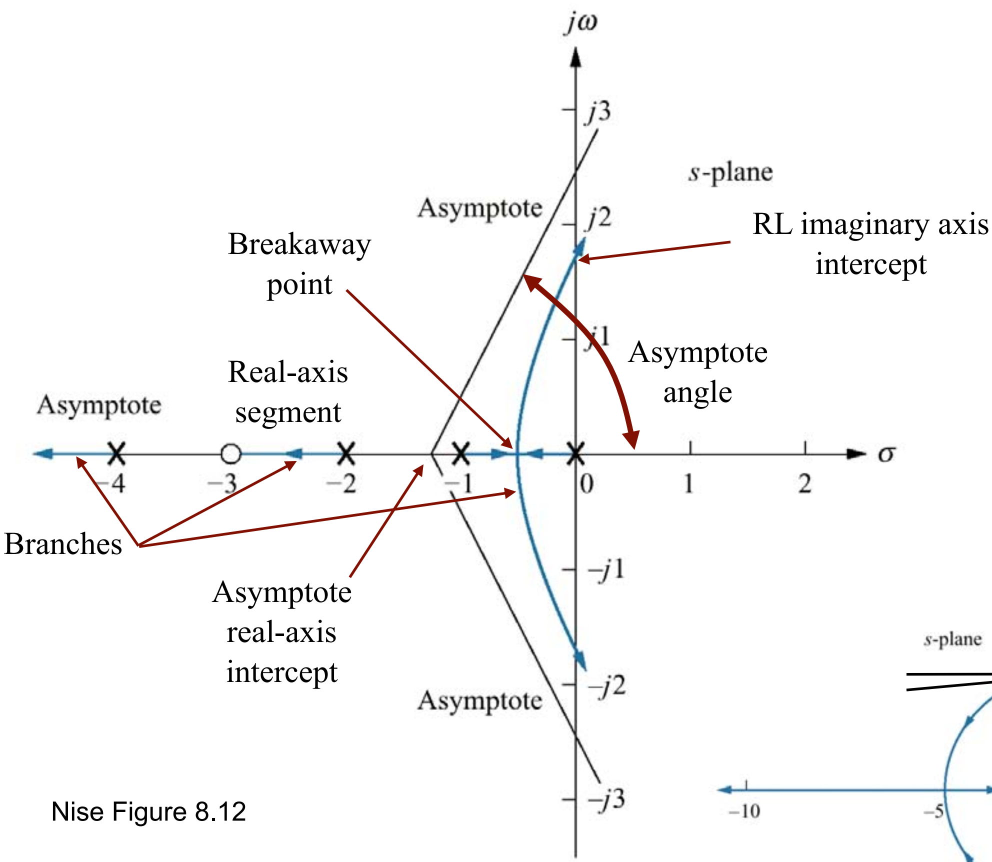


"Open loop" TF: $KG(s)$

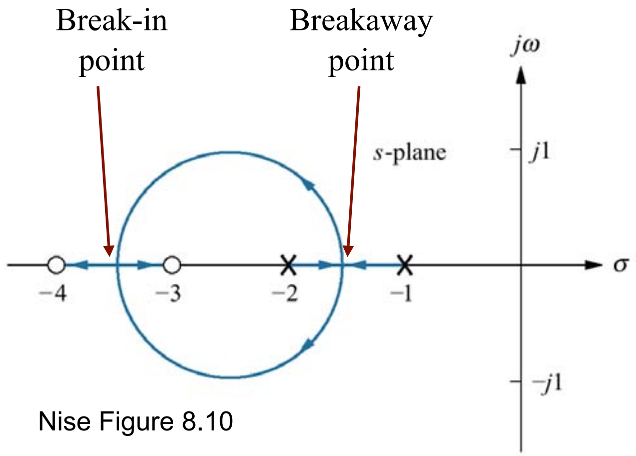
Condition for closed-loop pole: denominator of closed-loop TF must equal zero:

$$1 + KG(s)H(s) = 0 \Rightarrow \begin{cases} K = 1/\left|G(s)H(s)\right|; \\ \angle KG(s)H(s) = (2n+1)180^\circ. \end{cases}$$

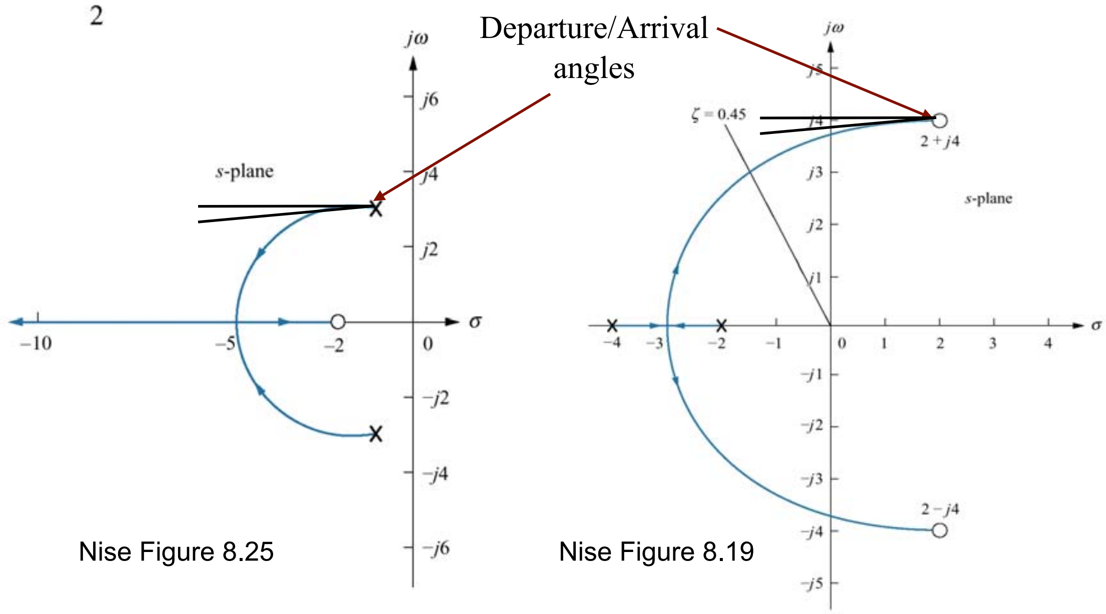
Root Locus terminology



Nise Figure 8.12



Nise Figure 8.10



Nise Figure 8.25

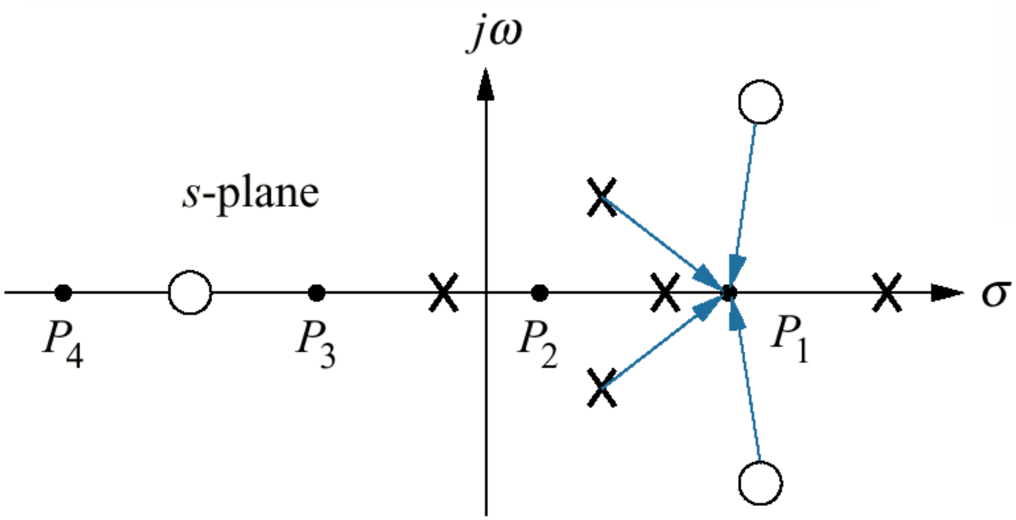
Nise Figure 8.19

Root-locus sketching rules

- **Rule 1: #** branches = # poles
- **Rule 2:** always symmetrical about the real axis
- **Rule 3:** real-axis segments are to the left of an *odd* number of real-axis finite poles/zeros

Let $G(s) = \dfrac{N_G(s)}{D_G(s)}, \quad H(s) = \dfrac{N_H(s)}{D_H(s)}.$

$$\Rightarrow \quad \angle G(s)H(s) = \sum \angle(\text{poles}) - \sum \angle(\text{zeros}).$$



Recall angle condition for closed–loop pole:

$$\angle KG(s)H(s) = (2n+1)180°.$$

Complex–pole/zero contributions: **cancel** because of symmetry

Real–pole/zero contributions: each is 0° from the left, 180° from the right; total contributions from right must be number of 180°'s to satisfy angle condition.

Root-locus sketching rules

- **Rule 4:** RL begins at poles, ends at zeros

$$\text{Let}\quad G(s)=\frac{N_G(s)}{D_G(s)},\quad H(s)=\frac{N_H(s)}{D_H(s)}.$$

$$\Rightarrow\quad \text{Closed–loop TF}(s)=\frac{KN_G(s)D_H(s)}{D_G(s)D_H(s)+KN_G(s)N_H(s)}.$$

If $K\to 0^+$ (small gain limit)

$$\text{Closed–loop TF}(s)\approx\frac{KN_G(s)D_H(s)}{D_G(s)D_H(s)+\epsilon}\Rightarrow$$

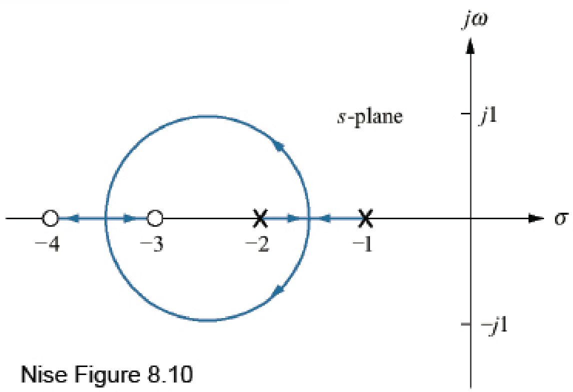
closed–loop denominator is *denominator* of $G(s)H(s)$
$\Rightarrow$closed–loop poles are the *poles* of $G(s)H(s)$.

If $K\to+\infty$ (large gain limit)

$$\text{Closed–loop TF}(s)\approx\frac{KN_G(s)D_H(s)}{\epsilon+KN_G(s)N_H(s)}\Rightarrow$$

closed–loop denominator is *numerator* of $G(s)H(s)$
$\Rightarrow$closed–loop poles are the *zeros* of $G(s)H(s)$.

Example



Nise Figure 8.10

Poles and zeros at infinity

$T(s)$ has a *zero at infinity* if $T(s \to \infty) \to 0$.

$T(s)$ has a *pole at infinity* if $T(s \to \infty) \to \infty$.

Example

$$KG(s)H(s) = \frac{K}{s(s+1)(s+2)}.$$

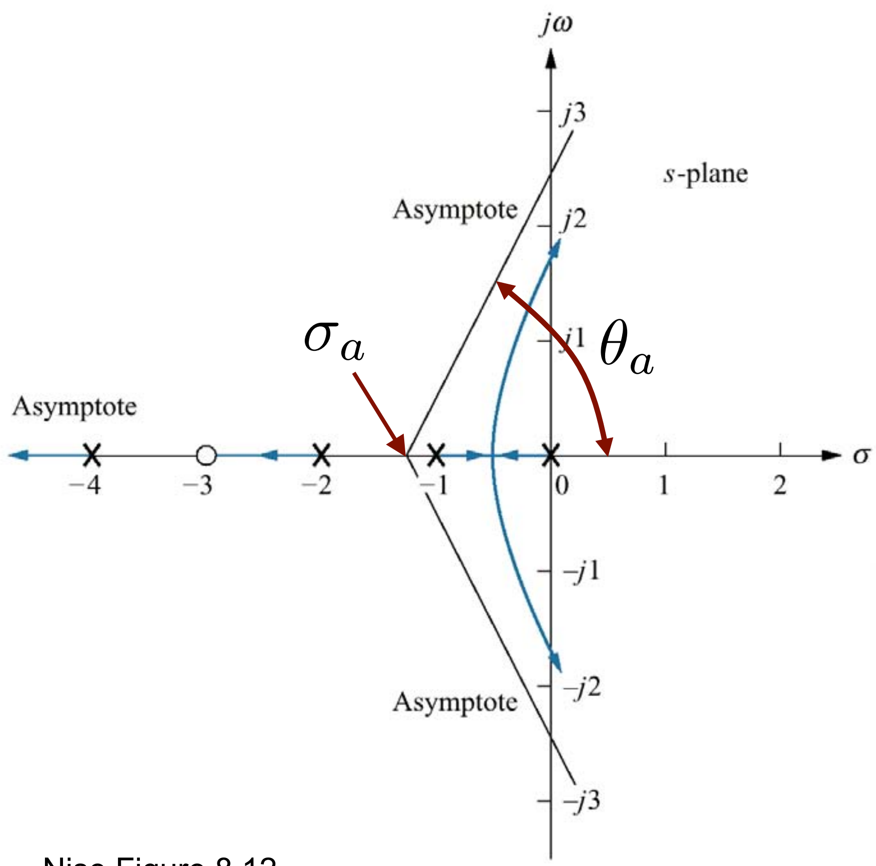
Clearly, this open–loop transfer function has three poles, 0, −1, −2. It has no *finite* zeros.
For large s, we can see that

$$KG(s)H(s) \approx \frac{K}{s^3}.$$

So this open–loop transfer function has **three** *zeros at infinity.*

Root Locus sketching rules

- **Rule 5:** Asymptotes: angles and real-axis intercept



Nise Figure 8.12

$$\sigma_a = \frac{\sum \text{finite poles} - \sum \text{finite zeros}}{\#\text{finite poles} - \#\text{finite zeros}}$$

$$\theta_a = \frac{(2m+1)\pi}{\#\text{finite poles} - \#\text{finite zeros}}$$

$$m = 0, \pm 1, \pm 2, \ldots$$

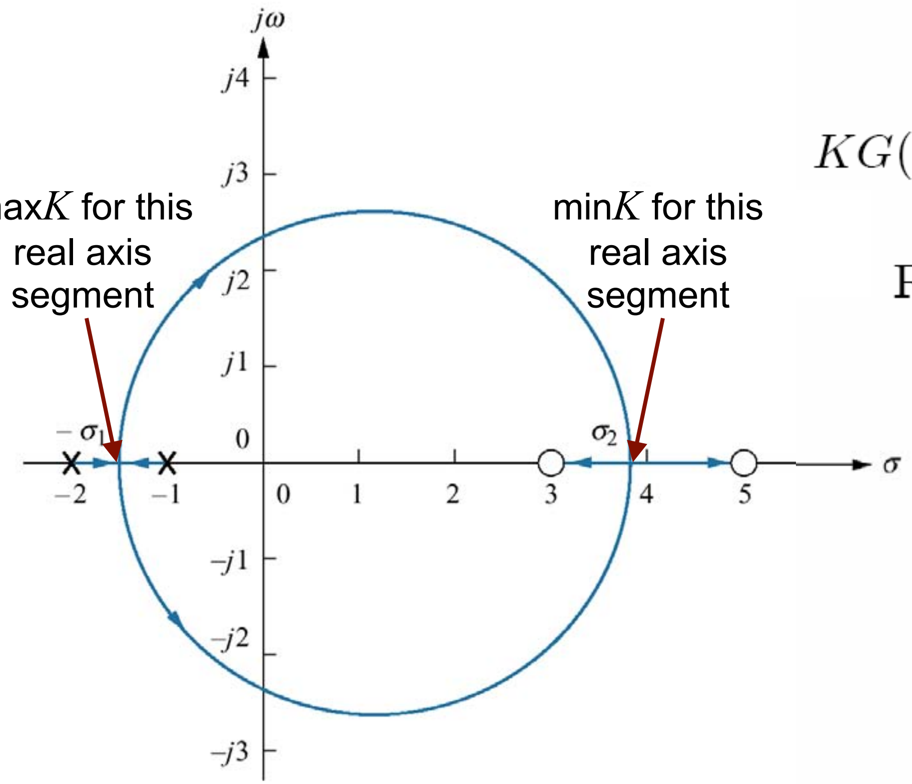
In this example, poles $= \{0, -1, -2, -4\}$, zeros $= \{-3\}$ so

$$\sigma_a = \frac{[0 + (-1) + (-2) + (-4)] - [(-3)]}{4-1} = -\frac{4}{3}$$

$$\theta_a = \frac{(2m+1)\pi}{4-1} = \left\{\frac{\pi}{3}, \pi, \frac{5\pi}{3}\right\}$$

Root Locus sketching rules

- **Rule 6:** Real axis break-in and breakaway points



Nise Figure 8.13

For each $s = \sigma$ on a real–axis segment of the root locus,

$$KG(\sigma)H(\sigma) = -1 \Rightarrow K = -\frac{1}{G(\sigma)H(\sigma)} \qquad (1)$$

Real–axis break–in & breakaway points are the real values of σ for which

$$\frac{\mathrm{d}K(\sigma)}{\mathrm{d}\sigma} = 0,$$

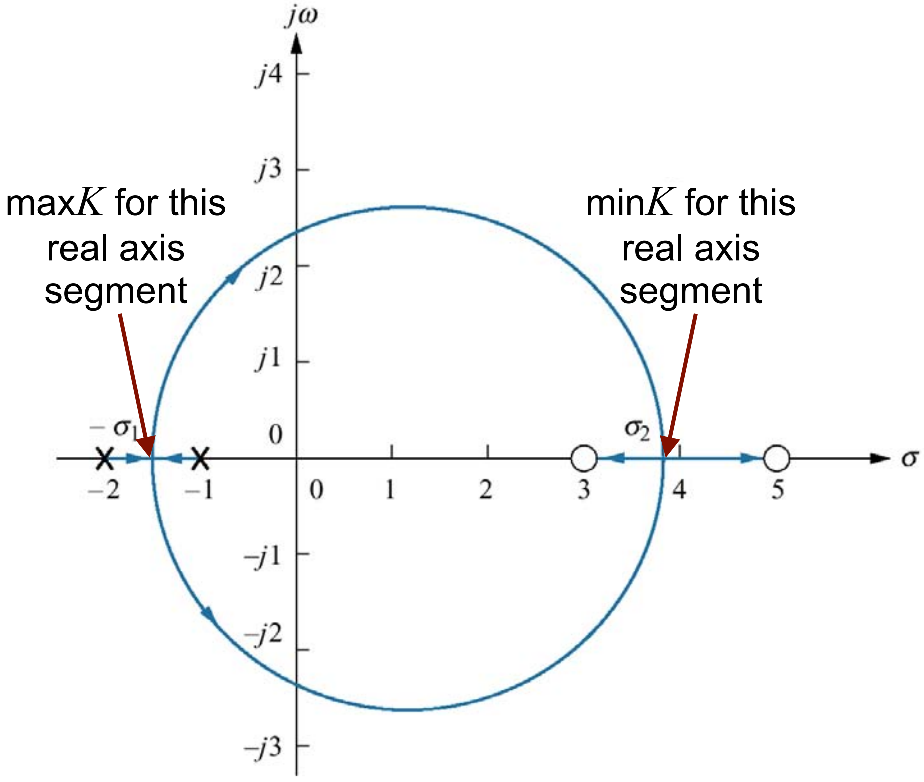
where $K(\sigma)$ is given by (1) above. Alternatively, we can solve

$$\sum \frac{1}{\sigma + z_i} = \sum \frac{1}{\sigma + p_i}.$$

for real σ.

Root Locus sketching rules

- **Rule 6:** Real axis break-in and breakaway points



Nise Figure 8.13

In this example,

$$KG(s)H(s) = \frac{K(s-3)(s-5)}{(s+1)(s+2)}$$

so on the real–axis segments we have

$$K(\sigma) = -\frac{(\sigma+1)(\sigma+2)}{(\sigma-3)(\sigma-5)} = -\frac{\sigma^2+3\sigma+2}{\sigma^2-8\sigma+15}$$

Taking the derivative,

$$\frac{\mathrm{d}K}{\mathrm{d}\sigma} = -\frac{11\sigma^2-26\sigma-61}{(\sigma^2-8\sigma+15)^2}$$

and setting $\mathrm{d}K/\mathrm{d}\sigma = 0$ we find

$$\sigma_1 = -1.45 \qquad \sigma_2 = 3.82$$

Alternatively, poles $= \{-1, -2\}$, zeros $= \{+3, +5\}$ so we must solve

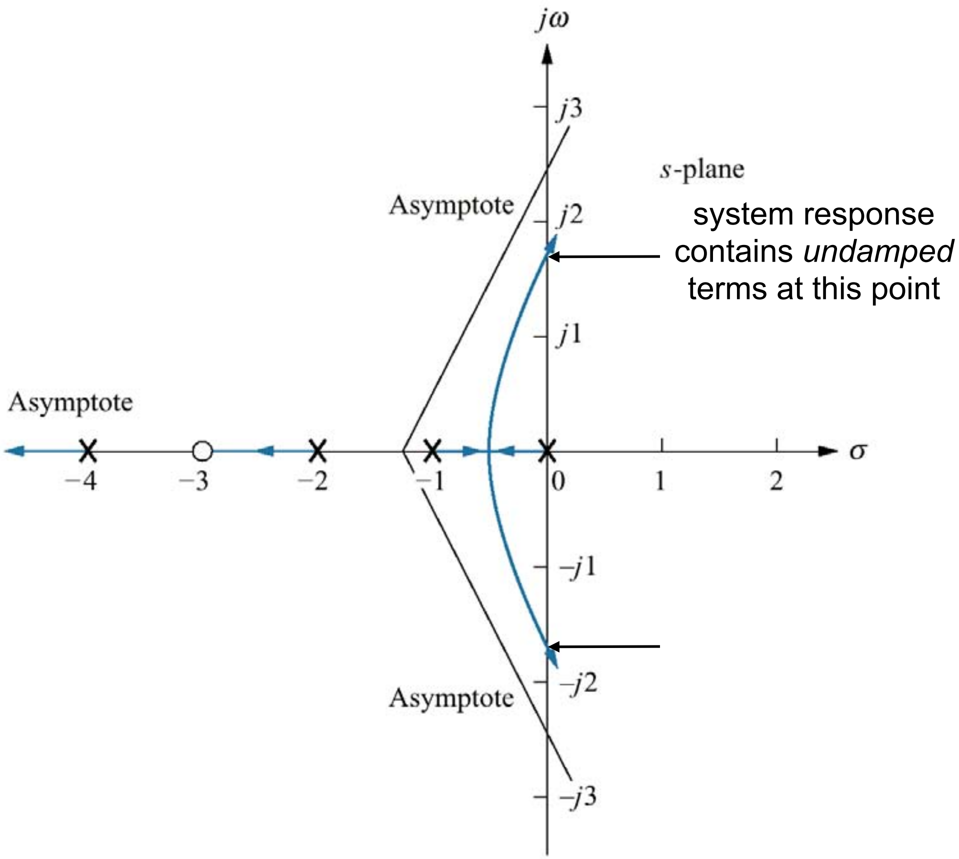
$$\frac{1}{\sigma-3} + \frac{1}{\sigma-5} = \frac{1}{\sigma+1} + \frac{1}{\sigma+2} \Rightarrow$$

$$11\sigma^2 - 26\sigma - 61 = 0.$$

This is the same equation as before.

Root Locus sketching rules

- **Rule 7:** Imaginary axis crossings



If $s = j\omega$ is a closed–loop pole on the imaginary axis, then

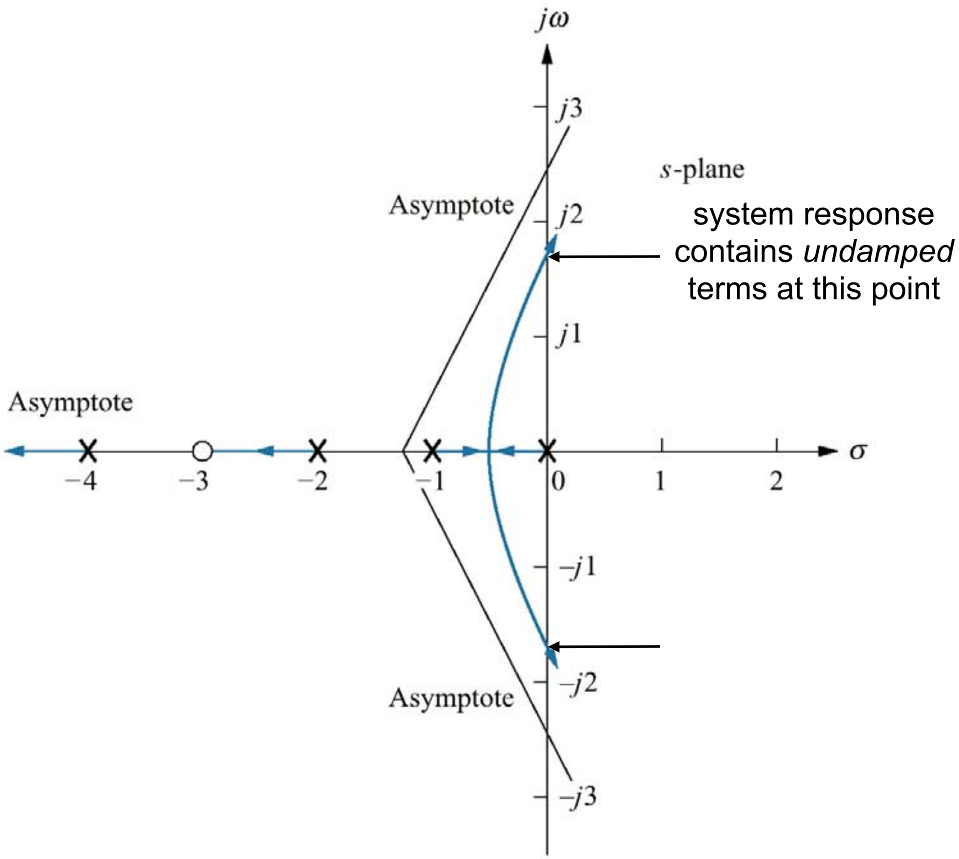
$$KG(j\omega)H(j\omega) = -1 \qquad (2)$$

The real and imaginary parts of (2) provide us with a 2×2 system of equations, which we can solve for the two unknowns K and ω (*i.e.*, the critical gain beyond which the system goes unstable, and the oscillation frequency at the critical gain.)

Note: Nise suggests using the Ruth–Hurwitz criterion for the same purpose. Since we did not cover Ruth–Hurwitz, we present here an alternative but just as effective method.

Root Locus sketching rules

- **Rule 7:** Imaginary axis crossings



In this example,

$$
\begin{aligned}
KG(s)H(s) &= \frac{K(s+3)}{s(s+1)(s+2)(s+4)} \\
&= \frac{Ks+3K}{s^4+7s^3+14s^2+8s} \Rightarrow \\
KG(j\omega)H(j\omega) &= \frac{jK\omega+3K}{\omega^4-j7\omega^3-14\omega^2+j8\omega}.
\end{aligned}
$$

Setting $KG(j\omega)H(j\omega)=-1$,

$$-\omega^4+j7\omega^3+14\omega^2-j(8+K)\omega-3K=0.$$

Separating real and imaginary parts,

$$\left\{\begin{aligned} -\omega^4+14\omega^2-3K &= 0, \\ 7\omega^3-(8+K)\omega &= 0. \end{aligned}\right.$$

In the second equation, we can discard the trivial solution $\omega=0$. It then yields

$$\omega^2=\frac{8+K}{7}.$$

Substituting into the first equation,

$$
\begin{aligned}
-\left(\frac{8+K}{7}\right)^2+14\left(\frac{8+K}{7}\right)-3K &= 0 \Rightarrow \\
K^2+65K-720 &= 0.
\end{aligned}
$$

Of the two solutions $K=-74.65$, $K=9.65$ we can discard the negative one (negative feedback $\Rightarrow K>0$). Thus, $K=9.65$ and $\omega=\sqrt{(8+9.65)/7}=1.59$.

Root Locus sketching rules summary

- **Rule 1:** # branches = # poles
- **Rule 2:** symmetrical about the real axis
- **Rule 3:** real-axis segments are to the left of an *odd* number of real-axis finite poles/zeros
- **Rule 4:** RL begins at poles, ends at zeros
- **Rule 5:** Asymptotes: angles, real-axis intercept

$$\sigma_a = \frac{\sum \text{finite poles} - \sum \text{finite zeros}}{\#\text{finite poles} - \#\text{finite zeros}} \qquad \theta_a = \frac{(2m+1)\pi}{\#\text{finite poles} - \#\text{finite zeros}} \qquad m = 0, \pm 1, \pm 2, \dots$$

- **Rule 6:** Real-axis break-in and breakaway points

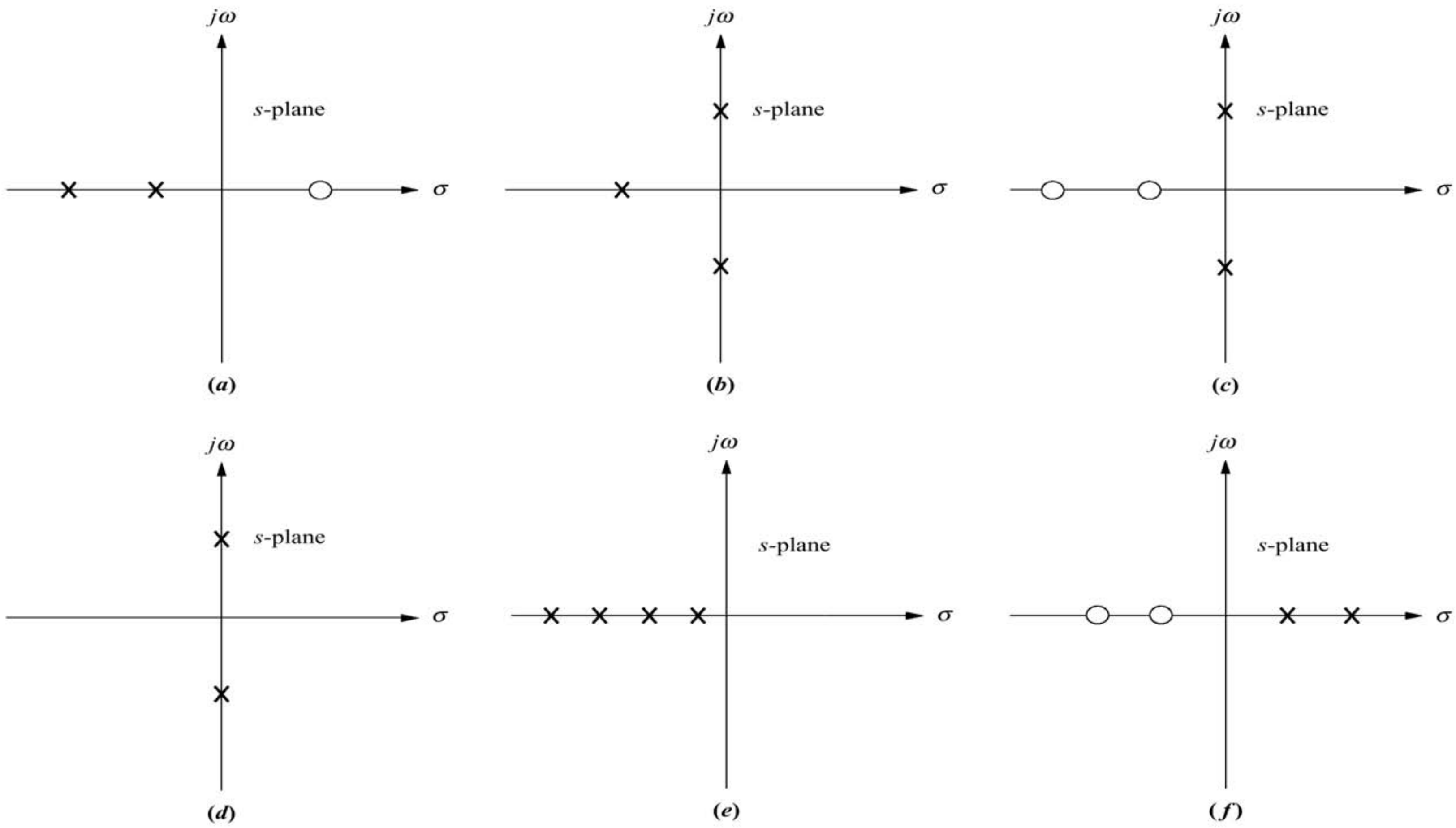
Found by setting $K(\sigma) = -\frac{1}{G(\sigma)H(\sigma)}$ (σ real) and solving $\frac{\mathrm{d}K(\sigma)}{\mathrm{d}\sigma} = 0$ for real σ.

- **Rule 7:** Imaginary axis crossings *(transition to instability)*

Found by setting $KG(j\omega)H(j\omega) = -1$ and solving

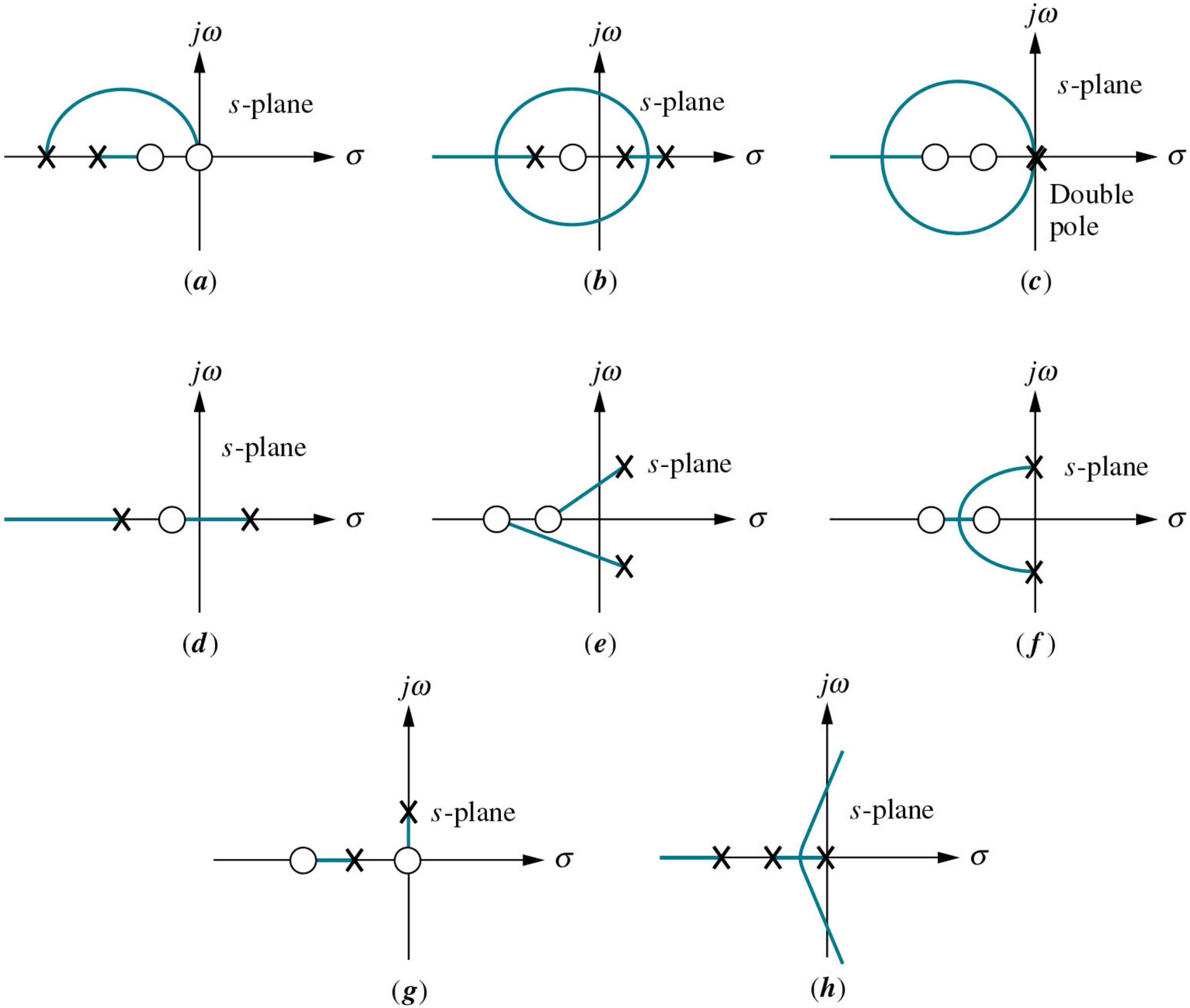
$$\begin{cases} \mathrm{Re}\left[KG(j\omega)H(j\omega)\right] = -1, \\ \mathrm{Im}\left[KG(j\omega)H(j\omega)\right] = 0. \end{cases}$$

Practice 1: Sketch the Root Locus



Nise Figure P8.2

Practice 2:
Are these Root Loci valid? If not, correct them



Nise Figure P8.1

In-class Experiment 4

- **What Is Root Locus Design?**

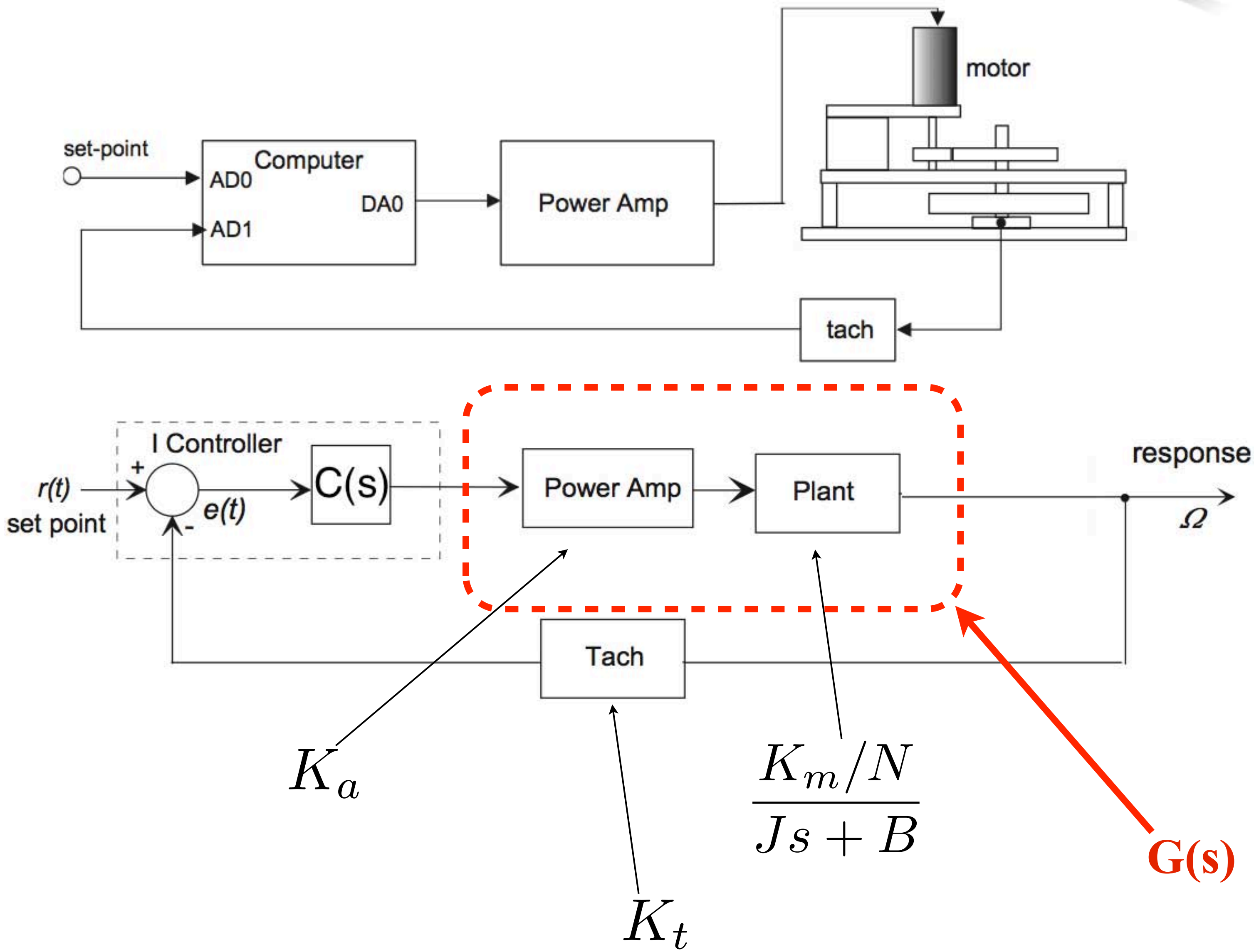
– A common technique involving iterating on a design by manipulating the compensator gain, poles, and zeros in the root locus diagram.

– As system parameter k varies over a range of values, the root locus diagram shows the trajectories of the closed-loop poles of the feedback system.

- **SISO Design Tool in MATLAB**:

– A graphical-user interface that allows the user to tune control parameters from root locus design and system response simulation.

System Modeling



System Parameters

- $J_{eq} = 0.03$ N-m^2.
- $B_{eq} = 0.014$ N-m-s/rad (lab average).
- $K_a = 2.0$ A/v.
- $K_m = 0.0292$ N-m/A (lab average).
- $K_t = (0.016 \frac{\text{v}}{\text{rev/min}})(60 \frac{\text{s}}{\text{min}})(\frac{1}{2\pi} \frac{\text{rev}}{\text{rad}}) = 0.153$ v/(rad/s).
- $N = \frac{44}{180} = 0.244$

Procedures

- In MATLAB workspace, construct necessary system data (transfer functions) based on the system model

- Graphically tune the control parameters of the following general forms.

$$C(s) = a_1, \quad C(s) = a_2 + b_2 s, \quad C(s) = a_3 + b_3/s$$

- At the end of the class, **turn in** your result parameters, root locus plots and system response.

Useful Matlab Commands

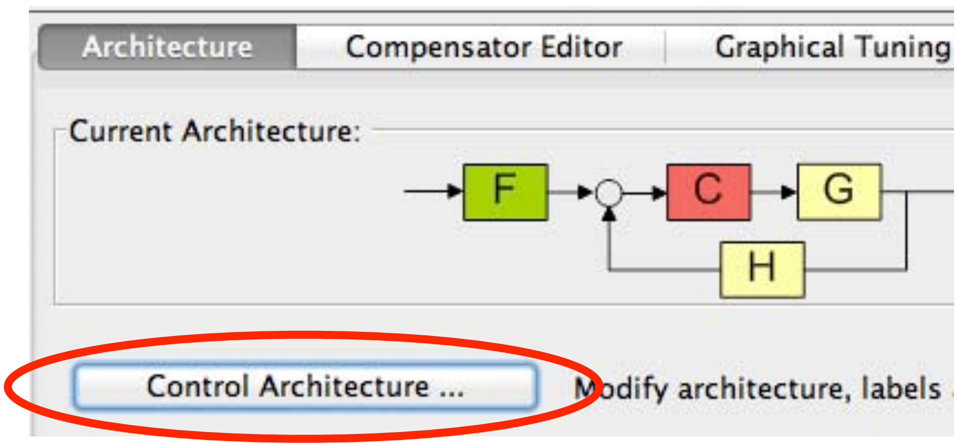
```
>> B=0.014;J=0.03;N=44/180;ka=2;km=0.0292;kt=0.016;
>> G=tf([ka*km/N],[J,B]);

Transfer function:

   0.2389
---------------
0.03 s + 0.0872
```

Setup transfer function

Tips 1

- In the command window type in “sisotool” tool open the SISO Design Tool Interface.

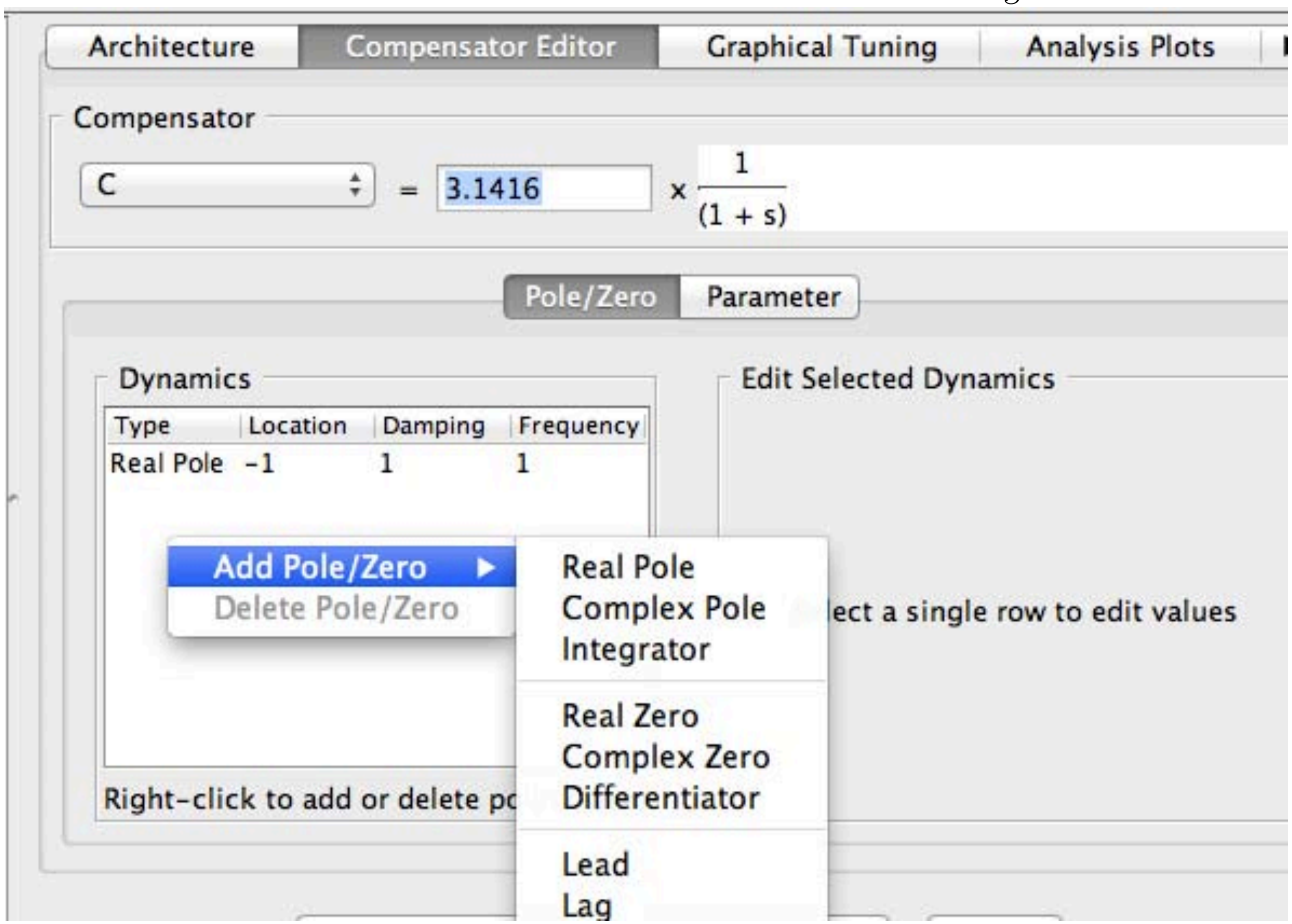
- Select appropriate control architecture



- Enter or import system **system data** (G, H) from workspace.

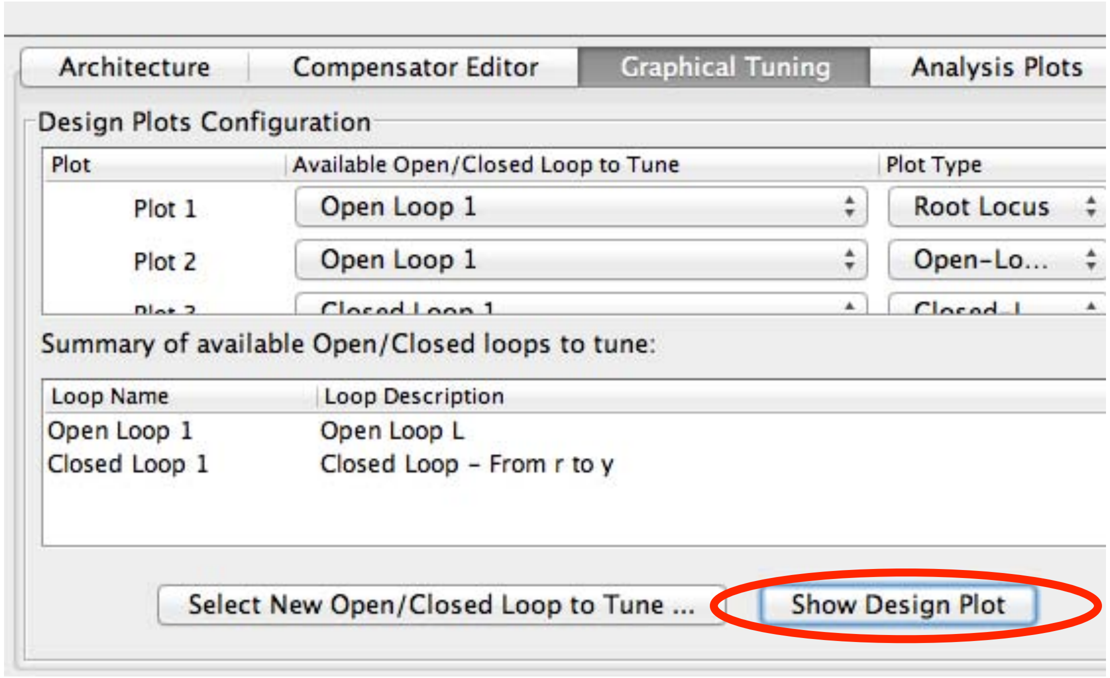
Tips 2

- Under "Compensator Editor" tab, create general form of controller model (e.g : $K_d s + K_p + K_i/s = K_d \frac{s^2 + \frac{K_p}{K_d} s + \frac{K_i}{K_d}}{s}$)



Tips 3

- Select design plots you want to use and click on **show design plot** under **Graphical Tuning**.



- You can **drag/add/remove** poles & zeros in this graphical root locus design window. Simulation result is instantaneous.

Tip4

- Select "Step" for Plot 1, "check closed loop r to y". **Show analysis plot** under Analysis Plot tab generates a real-time step response of your system. (You can also look at other plots)