2.032 Dynamics

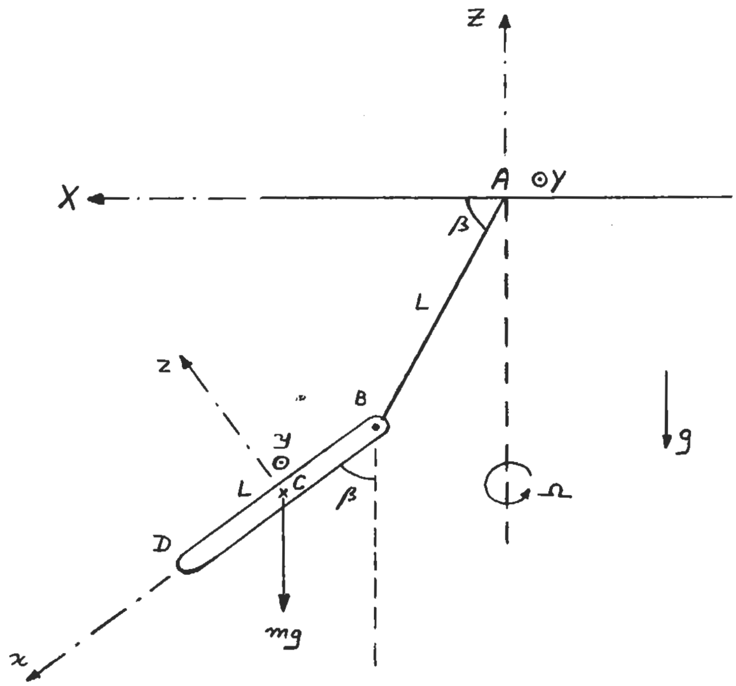
Fall 2003

Quiz No. 2

Problem 1

Both XYZ and xyz rotate with Ω about Z.

The bar and the cable are always in the plane of XZ or xz.



$$\underset{\sim}{AB} = L\cos\beta\, \underset{\sim}{\hat{e}}_X - L\sin\beta\, \underset{\sim}{\hat{e}}_Z$$

$$\underset{\sim}{\omega}|_{AB} = \Omega\, \underset{\sim}{\hat{e}}_Z$$

$$\underset{\sim}{v}_B = \frac{d}{dt}\underset{\sim}{AB} = \underset{\sim}{\omega}|_{AB} \times \underset{\sim}{AB} = \Omega L\cos\beta\, \underset{\sim}{\hat{e}}_Y$$

$$\underset{\sim}{BC} = \frac{L}{2}\sin\beta\, \underset{\sim}{\hat{e}}_X - \frac{L}{2}\cos\beta\, \underset{\sim}{\hat{e}}_Z$$

$$\underset{\sim}{\omega}|_{BC} = \Omega\, \underset{\sim}{\hat{e}}_Z$$

$$\frac{d}{dt}\underset{\sim}{BC} = \underset{\sim}{\omega}|_{BC} \times \underset{\sim}{BC} = \Omega \frac{L}{2}\sin\beta\, \underset{\sim}{\hat{e}}_Y$$

$$\underset{\sim}{v}_C = \frac{d}{dt}\underset{\sim}{AC} = \frac{d}{dt}\left(\underset{\sim}{AB} + \underset{\sim}{BC}\right) = \Omega L\left(\cos\beta + \frac{\sin\beta}{2}\right)\underset{\sim}{\hat{e}}_Y$$

Angular momentum about point B:

$$\underset{\sim}{\tau}_B = \frac{d}{dt}\underset{\sim}{H}_B + \overbrace{\underset{\sim}{v}_B \times m\underset{\sim}{v}_C}^{0} = \frac{d}{dt}\underset{\sim}{H}_B$$

Problem 1

$$\underset{\sim}{H}_B = \underset{\sim}{H}_C + \underset{\sim}{BC} \times m\underset{\sim}{v}_C$$

$$\underset{\sim}{\omega}\Big|_{bar} = \Omega\, \underset{\sim}{\hat{e}}_Z = -\Omega \cos\beta\, \underset{\sim}{\hat{e}}_x + \Omega \sin\beta\, \underset{\sim}{\hat{e}}_z$$

slender bar

$$\underset{\sim}{H}_C = [I]_C\, \underset{\sim}{\omega}\Big|_{bar} = \frac{1}{12} m L^2 \begin{bmatrix} \approx 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} -\Omega \cos\beta \\ 0 \\ \Omega \sin\beta \end{Bmatrix} = \frac{1}{12} m L^2 \Omega \sin\beta\, \underset{\sim}{\hat{e}}_z \qquad (I_x \approx 0 \text{ for slender bar})$$

$$\left(\underset{\sim}{\hat{e}}_z = \cos\beta\, \underset{\sim}{\hat{e}}_X + \sin\beta\, \underset{\sim}{\hat{e}}_Z \right)$$

$$\therefore \quad \underset{\sim}{H}_B = \frac{1}{12} m L^2 \Omega \sin\beta \left(\cos\beta\, \underset{\sim}{\hat{e}}_X + \sin\beta\, \underset{\sim}{\hat{e}}_Z \right) + m \left(\frac{L}{2} \sin\beta\, \underset{\sim}{\hat{e}}_X - \frac{L}{2} \cos\beta\, \underset{\sim}{\hat{e}}_Z \right) \times \Omega L \left(\cos\beta + \frac{\sin\beta}{2} \right) \underset{\sim}{\hat{e}}_y$$

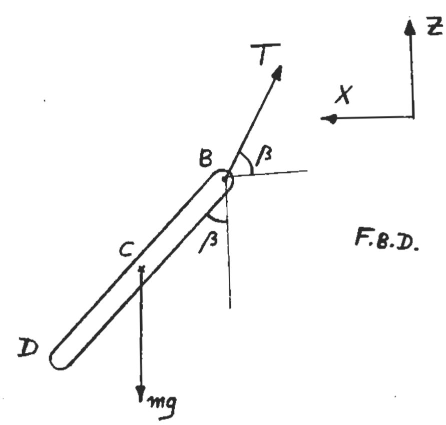
$$\longrightarrow \quad \underset{\sim}{H}_B = \left[\frac{1}{3} m L^2 \Omega \sin\beta \cos\beta + m\Omega \frac{L^2}{2} \cos^2\beta \right] \underset{\sim}{\hat{e}}_X + \left[\frac{1}{3} m L^2 \Omega \sin^2\beta + m\Omega \frac{L^2}{2} \sin\beta \cos\beta \right] \underset{\sim}{\hat{e}}_Z$$

$$\frac{d}{dt} \underset{\sim}{H}_B = m L^2 \Omega \cos\beta \left(\frac{\sin\beta}{3} + \frac{\cos\beta}{2} \right) \frac{d}{dt} \underset{\sim}{\hat{e}}_X \longrightarrow \Omega\, \underset{\sim}{\hat{e}}_Z \times \underset{\sim}{\hat{e}}_X = \Omega\, \underset{\sim}{\hat{e}}_y$$

$$\underset{\sim}{\tau}_B = mg \frac{L}{2} \sin\beta\, \underset{\sim}{\hat{e}}_y$$

$$\Longrightarrow \quad mg \frac{L}{2} \sin\beta = m L^2 \Omega^2 \cos\beta \left(\frac{\sin\beta}{3} + \frac{\cos\beta}{2} \right)$$

$$\Longrightarrow \quad \boxed{\Omega^2 = \frac{g \tan\beta}{L \left(\frac{2}{3} \sin\beta + \cos\beta \right)}} \qquad (*)$$



F.B.D.

Linear momentum:

$$\underset{\sim}{F}_{ext} = \frac{d}{dt} m \underset{\sim}{v}_C$$

$$\underset{\sim}{F}_{ext} = (T \sin\beta - mg)\, \underset{\sim}{\hat{e}}_Z - T \cos\beta\, \underset{\sim}{\hat{e}}_X$$

$$\frac{d}{dt} m \underset{\sim}{v}_C = m \Omega L \left(\cos\beta + \frac{\sin\beta}{2} \right) \frac{d}{dt} \underset{\sim}{\hat{e}}_y \longrightarrow \Omega\, \underset{\sim}{\hat{e}}_Z \times \underset{\sim}{\hat{e}}_y = -\Omega\, \underset{\sim}{\hat{e}}_X$$

Problem 1

$$\therefore \quad \begin{cases} T\sin\beta = mg \\ -mL\Omega^2\left(\cos\beta + \frac{\sin\beta}{2}\right) = -T\cos\beta \end{cases}$$

Eliminating T :

$$\boxed{\Omega^2 = \frac{g}{L\tan\beta\left(\cos\beta + \frac{\sin\beta}{2}\right)}} \qquad (**)$$

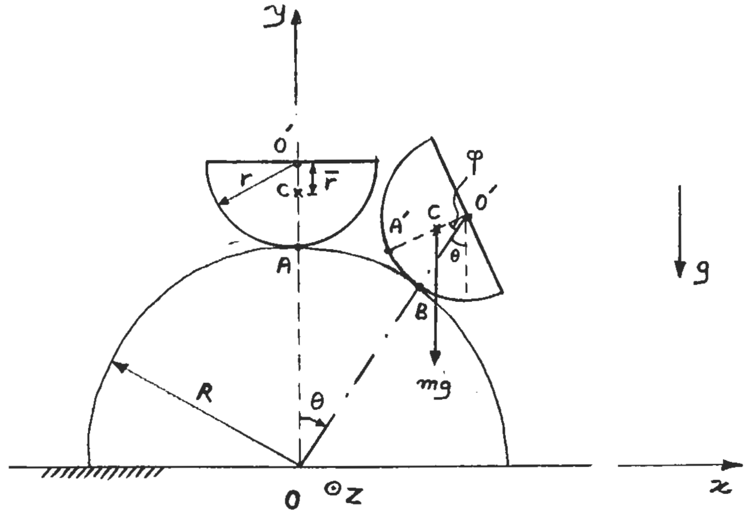
$$(*),(**) \implies \frac{1}{2}\tan^3\beta + \tan^2\beta - \frac{2}{3}\tan\beta - 1 = 0 \implies \begin{cases} \tan\beta = 1.056 \\ \cancel{\tan\beta = -2.192} \\ \cancel{\tan\beta = -0.864} \end{cases} \qquad \left(0 < \beta < \frac{\pi}{2}\right)$$

$$\tan\beta = 1.056 \xrightarrow{(*)} \Omega^2 = 0.901\,\frac{g}{L} \longrightarrow \Omega = 0.949\sqrt{\frac{g}{L}}$$

$$\boxed{\begin{aligned} \beta &= \tan^{-1}(1.056) \\ \Omega &= 0.949\sqrt{\frac{g}{L}} \end{aligned}}$$

Problem 2

The system has one degree of freedom so angle θ specifies position of half cylinder completely.



(C : center of mass of the half cylindr)

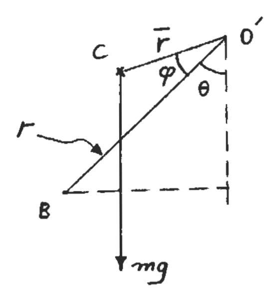
No slip : $BA' = BA \longrightarrow r\varphi = R\theta \longrightarrow \varphi = \frac{R}{r}\theta$

Obviously, the equilibrium is at $\theta = 0$ where the torque about contact point A is zero.

To check the stability to small perturbations, one can examine the torque about contact point B :

$$\underset{\sim}{\tau}_B = -mg\left[r\sin\theta - \bar{r}\sin(\theta+\varphi)\right]\underset{\sim}{\hat{e}}_z$$

$$\underset{\sim}{\tau}_B = -mg\left[r\sin\theta - \bar{r}\sin\left(1+\frac{R}{r}\right)\theta\right]\underset{\sim}{\hat{e}}_z = \tau_B\,\underset{\sim}{\hat{e}}_z$$



If $\tau_B < 0 \longrightarrow$ small perturbation would cause the half cylinder to roll down so the motion is away from the equilibrium $(\theta = 0)$. and the equilibrium is unstable. On the other hand, if $\tau_B > 0 \longrightarrow$ small perturbation would cause a motion toward equilibrium $(\theta = 0)$ and the half cylinder oscillates about the equilibrium so the system is stable.

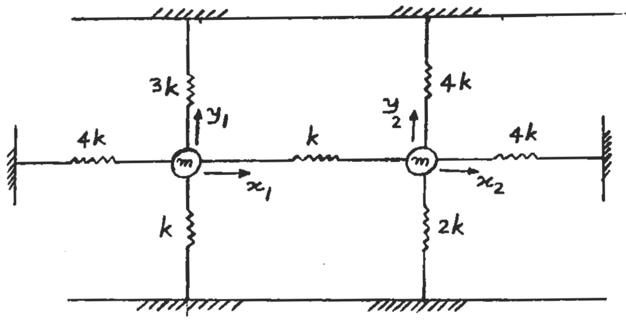
Therefore, condition for stability of the equilibrium point :

$$\tau_B > 0 \longrightarrow -mg\left[r\sin\theta - \bar{r}\sin\left(1+\frac{R}{r}\right)\theta\right] > 0$$

$$\theta \text{ small} \longrightarrow r\theta - \bar{r}\left(1+\frac{R}{r}\right)\theta < 0 \xrightarrow{\theta > 0} \bar{r} > \frac{r^2}{r+R} \overset{\bar{r} = \frac{4r}{3\pi}}{\Longrightarrow} \frac{r}{R} < \frac{4}{3\pi - 4}$$

Problem 3

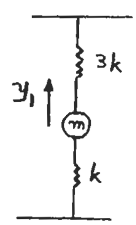
The system has four degrees of freedom so it has four natural frequencies and modes.

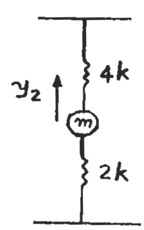


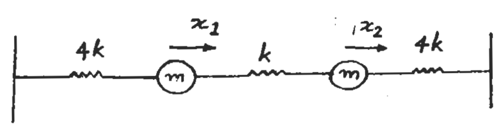
For small departures from equilibrium, horizontal and vertical motions are uncoupled since in the horizontal (vertical) motion, vertical (horizontal) springs do not play a part. Furthermore, the two vertical motions are uncoupled. This can also be easily shown by using either Lagrangian or direct method.

So, we have the following uncoupled systems:

$$\{x\} = \begin{Bmatrix} x_1 \\ x_2 \\ y_1 \\ y_2 \end{Bmatrix}$$







Now it is easy to guess the modes:

Vertical motion of the mass on the left :

$$\{a\}_1 = \begin{Bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{Bmatrix}, \quad \omega_1^2 = \frac{k+3k}{m} = \frac{4k}{m}$$

Vertical motion of the mass on the right :

$$\{a\}_2 = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{Bmatrix}, \quad \omega_2^2 = \frac{2k+4k}{m} = \frac{6k}{m}$$

Problem 3

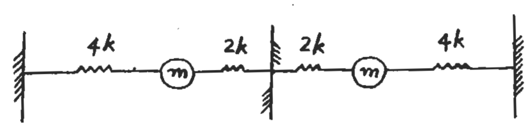
For the horizontal motion,

$$\{a\}_3 = \begin{Bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{Bmatrix}, \qquad \omega_3^2 = \frac{4k}{m}$$

In this mode, the spring in the middle does not come into play ($x_1 = x_2$).

$$\{a\}_4 = \begin{Bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{Bmatrix}, \qquad \omega_4^2 = \frac{6k}{m}$$

In this mode, the equivalent system would be:



Because of symmetry, center of the spring in the middle does not move.