6 WATER WAVES

Surface waves in water are a superb example of a stationary and ergodic random process. The model of waves as a nearly linear superposition of harmonic components, at random phase, is confirmed by measurements at sea, as well as by the linear theory of waves, the subject of this section.

We will skip some elements of fluid mechanics where appropriate, and move quickly to the cases of two-dimensional, inviscid and irrotational flow. These are the major assumptions that enable the linear wave model.

6.1 Constitutive and Governing Relations

First, we know that near the sea surface, water can be considered as incompressible, and that the density ρ is nearly uniform. In this case, a simple form of conservation of mass will hold:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0,$$

where the Cartesian space is $[x, y, z]$, with respective particle velocity vectors $[u, v, w]$. In words, the above equation says that net flow into a differential volume has to equal net flow out of it. Considering a box of dimensions $[\delta x, \delta y, \delta z]$, we see that any δu across the x-dimension, has to be accounted for by δv and δw:

$$\delta u \delta y \delta z + \delta v \delta x \delta z + \delta w \delta x \delta y = 0.$$
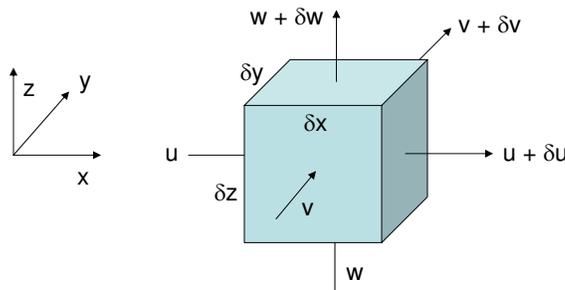
Next, we invoke Newton's law, in the three directions:

$$\rho \left[\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} \right] = -\frac{\partial p}{\partial x} + \mu \left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right];$$

$$\rho \left[\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} + w\frac{\partial v}{\partial y} + u\frac{\partial v}{\partial z} \right] = -\frac{\partial p}{\partial y} + \mu \left[\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} \right];$$

$$\rho \left[\frac{\partial w}{\partial t} + w\frac{\partial w}{\partial x} + u\frac{\partial w}{\partial y} + v\frac{\partial w}{\partial z} \right] = -\frac{\partial p}{\partial z} + \mu \left[\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2} \right] - \rho g.$$

Here the left-hand side of each equation is the acceleration of the fluid particle, as it moves through the differential volume. The terms such as $u\frac{\partial u}{\partial x}$ capture the fact that the force balance is for a moving particle; the chain rule expansion goes like this in the x-direction:

$$\frac{du}{dt} = \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial t},$$

where $u = \partial x/\partial t$ and so on.

On the right side of the three force balance equations above, the differential pressure clearly acts to slow the particle (hence the negative sign), and viscous friction is applied through absolute viscosity μ. The third equation also has a term for gravity, leading in the case of zero velocities to the familiar relation $p(z) = -\rho g z$, where z is taken positive upward from the mean free surface.

6.2   Rotation and Viscous Effects

In a fluid, unlike for rigid bodies, rotation angle is taken as the average of the angular deflections of the faces. Hence, a net rotation only occurs if the deflection of each face is additive. If they are opposing, then we have only a shearing of the element, with no rotation. Several cases are illustrated below for the two-dimensional case.
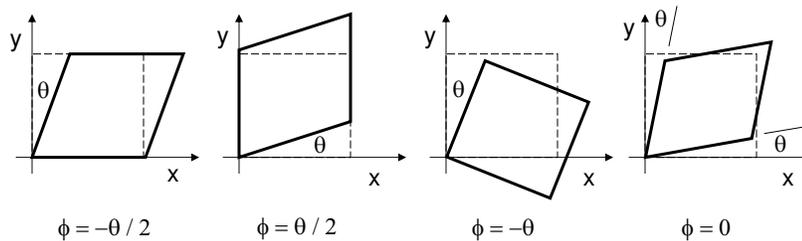
Now the rotation rate in the z direction is $\frac{\partial v}{\partial x}$ (the counterclockwise deflection rate of the horizontal face in the plots above), minus $\frac{\partial u}{\partial y}$ (clockwise deflection rate of the vertical face in the plots above). Giving the three dimensional rotation rate vector symbol $\vec{\omega}$, we have

$$\vec{\omega} = \left[\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z} \quad \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x} \quad \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right]^T.$$

Despite this attention, we will now argue that rotational effects are negligible in large water waves. The center of the argument is the fact that a spherical particle will have no rotation except through shear forces. At the same time, however, the Reynolds number in ocean-scale waves has to be taken into account; it is the ratio of inertial to viscous forces

$$Re = \frac{Ud}{\nu},$$

where characteristic speed and length scales are U and d respectively, with ν the kinematic viscosity $(\mu\rho)$. The kinematic viscosity of water at typical ocean temperatures is $1e-6m^2/s$. In contrast, velocities encountered in ocean waves are on the order of $10m/s$, with flow structures on the scale of meters or more. Hence the Reynolds number is very large, and the viscous forces may be neglected. This means in particular that $\vec{\omega}$ is zero and that we will neglect all terms with μ in the force balance.

Note that the inviscid and irrotational assumption is not necessarily valid near solid boundaries, where very small flow structures associated with turbulence result from the no-slip boundary condition.

6.3   Velocity Potential

We introduce the vector field $\phi(\vec{x}, t)$ to satisfy the following relation:

$$\vec{V} = \left\{ \begin{array}{c} u \\ v \\ w \end{array} \right\} = \left[\begin{array}{ccc} \dfrac{\partial\phi}{\partial x} & \dfrac{\partial\phi}{\partial y} & \dfrac{\partial\phi}{\partial z} \end{array} \right]^T = \nabla\phi.$$

The conservation of mass is transformed to

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial x^y} + \frac{\partial^2\phi}{\partial z^2} = \nabla^2 \cdot \phi = 0.$$

Considering Newton's law, the first force balance (x-direction) that we gave above is

$$\rho\left[\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} \right] = -\frac{\partial p}{\partial x};$$

this becomes, substituting the velocity potential ϕ,

$$\rho\left[\frac{\partial^2\phi}{\partial t\partial x} + \frac{\partial\phi}{\partial x}\frac{\partial^2\phi}{\partial x^2} + \frac{\partial\phi}{\partial y}\frac{\partial^2\phi}{\partial y\partial x} + \frac{\partial\phi}{\partial z}\frac{\partial^2\phi}{\partial z\partial x} \right] = -\frac{\partial p}{\partial x}.$$

Integrating on x we find

$$p + \rho\frac{\partial\phi}{\partial t} + \frac{1}{2}\rho\left(u^2 + v^2 + w^2 \right) = C,$$

where C is a constant. The other two force balance equations are precisely the same but with the addition of gravity effects in the z-direction. Hence a single equation for the whole field is

$$p + \rho\frac{\partial\phi}{\partial t} + \frac{1}{2}\rho\left(u^2 + v^2 + w^2 \right) + \rho gz = C.$$

This is the Bernoulli equation.

6.4   Linear Waves

We consider small amplitude waves in two dimensions x and z, and call the surface deflection $\eta(x, t)$, positive in the positive z direction. Within the fluid, conservation of mass holds, and on the surface we will assume that $p = p_a$, the atmospheric pressure, which we can take to be zero since it is merely an offset to all the pressures under the surface. At the seafloor of course $w = 0$ because flow cannot move into and out of the boundary; at the same time, because of the irrotational flow assumption, the velocity u can be nonzero right to the boundary.
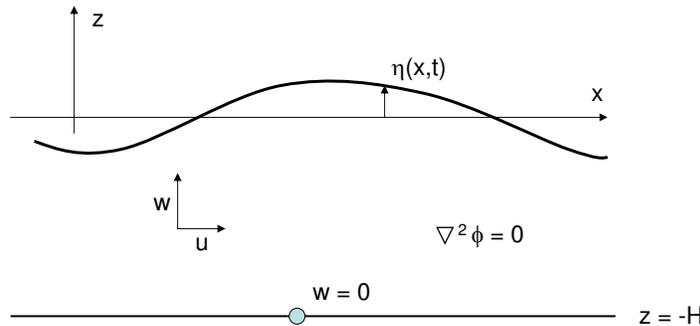
Taking a look at the relative size of terms, we see that $(u^2 + v^2 + w^2)/2$ is much smaller than gz - consider that waves have frequency of one radian per second or so, with characteristic size scale one meter (vertical), whereas g is of course order ten. Hence the Bernoulli equation near the surface simplifies to:

$$\rho \frac{\partial \phi}{\partial t} + \rho g \eta \approx 0 \text{ at } z = 0.$$

Next, we note the simple fact from our definitions that

$$\frac{\partial \eta}{\partial t} \approx \frac{\partial \phi}{\partial z} \text{ at } z = 0.$$

In words, the time rate of change of the surface elevation is the same as the z-derivative of the potential, namely w. Combining these two equations we obtain

$$\frac{\partial^2 \phi}{\partial t^2} + g \frac{\partial \phi}{\partial z} = 0 \text{ at } z = 0.$$

The solution for the surface is a traveling wave

$$\eta(x, t) = a \cos(\omega t - kx + \psi),$$

where a is amplitude, ω is the frequency, k is the wavenumber (see below), and ψ is a random phase angle. The traveling wave has speed ω/k. The corresponding candidate potential is

$$
\begin{aligned}
\phi(x, z, t) &= -\frac{a\omega}{k} \frac{\cosh(k(z+H))}{\sinh(kH)} \sin(\omega t - kx + \psi), \text{ where} \\
\omega &= 2\pi/T = \sqrt{kg \tanh(kH)} \text{ (dispersion)}, \\
k &= 2\pi/\lambda.
\end{aligned}
$$

Here λ is the wavelength, the horizontal extent between crests. Let us confirm that this potential satisfies the requirements. First, does it solve Bernoulli's equation at $z = 0$?

$$
\begin{aligned}
\frac{\partial^2 \phi}{\partial t^2} &= \frac{a\omega^3}{k} \frac{1}{\tanh kH} \sin(\omega t - kx + \psi) \\
&= a\omega g \sin(\omega t - kx + \psi) \text{ and} \\
\frac{\partial \phi}{\partial z} &= -a\omega \sin(\omega t - kx + \psi).
\end{aligned}
$$

Clearly Bernoulli's equation at the surface is satisfied. Working with the various definitions, we have further

$$
\begin{aligned}
u(x, z, t) &= \frac{\partial \phi}{\partial x} = a\omega \frac{\cosh(k(z + H))}{\sinh(kH)} \cos(\omega t - kx + \psi), \\
w(x, z, t) &= \frac{\partial \phi}{\partial z} = -a\omega \frac{\sinh(k(z + H))}{\sinh(kH)} \sin(\omega t - kx + \psi) \\
p(x, z, t) &\approx -\rho \frac{\partial \phi}{\partial t} - \rho g z \\
&= \rho \frac{a\omega^2}{k} \frac{\cosh(k(z + H))}{\sinh(kH)} \cos(\omega t - kx + \psi) - \rho g z.
\end{aligned}
$$

At the surface, $z = 0$, it is clear that the hyperbolic sines in $w(x, z, t)$ cancel. Then taking an integral on time easily recovers the expression given above for surface deflection $\eta(x, t)$. The pressure here is $\rho g \eta$, as would be expected. At depth $z = -H$, $w = 0$ because $\sinh(0) = 0$, thus meeting the bottom boundary condition. The particle trajectories in the x-direction and the z-direction are respectively

$$
\begin{aligned}
\xi_p(x, z, t) &= a \frac{\cosh(k(z + H))}{\sinh(kH)} \sin(\omega t - kx + \psi) \\
\eta_p(x, z, t) &= \frac{a}{k} \frac{\cosh(k(z + H))}{\sinh(kH)} \cos(\omega t - kx + \psi).
\end{aligned}
$$

Hence the particles' motions take the form of ellipses, clockwise when the wave is moving in the positive x direction.

Note that there are no nonlinear terms in $[x, y, z, u, v, w, p, \phi]$ in any of these equations, and hence this model for waves is linear. In particular, this means that waves of different frequencies and phases can be superimposed, without changing the behavior of the independent waves.

6.5   Deepwater Waves

In the limit that $H \longrightarrow \infty$, the above equations simplify because

$$
\phi(x, z, t) \longrightarrow -\frac{a\omega}{k} e^{kz} \sin(\omega t - kx + \psi).
$$

We find that

$$
\begin{aligned}
\omega^2 &= kg \text{ (dispersion)} \\
p &= \rho g a e^{kz} \cos(\omega t - kx + \psi) - \rho g z; \\
u &= a\omega e^{kz} \cos(\omega t - kx + \psi); \\
w &= -a\omega e^{kz} \sin(\omega t - kx + \psi); \\
\xi_p &= a e^{kz} \sin(\omega t - kx + \psi); \\
\eta_p &= a e^{kz} \cos(\omega t - kx + \psi).
\end{aligned}
$$

The dynamic part of the pressure undergoes an exponential decay in amplitude with depth. This is governed by the wave number k, so that the dynamic pressure is quite low below even one-half wavelength in depth: the factor is $e^{-\pi} \approx 0.05$. Particle motions become circular for the deepwater case. The radii of the circles also decay exponentially with depth.

6.6   Wave Loading of Stationary and Moving Bodies

The elegance of the linear wave theory permits explicit estimation of wave loads on structures, usually providing reasonable first approximations. We break the forces on the body into three classes:

1. The dynamic pressure load integrated over the body surface, with the assumption that the presence of the body does not affect the flow - it is a "ghost" body. We call this the incident wave force.

2. The flow is deflected from its course because of the presence of the body; assuming here that the body is stationary. This is the diffraction wave force.

3. Forces are created on the body by its moving relative still water. This is wavemaking due to the body pushing fluid out of the way. We call this the radiation wave force.

This separation of effects clearly depends on linearizing assumptions. Namely, the moving flow interacts with a stationary body in the incident wave and diffraction forces, whereas the stationary flow interacts with a moving body in the radiation force. Further, among the first two forces, we decompose into a part that is unaffected by the "ghost" body and a part that exists only because of the body's presence.

Without proof, we will state simple formulas for the diffraction and radiation loads, and then go into more detail on the incident wave (pressure) force.

As a prerequisite, we need the concept of added mass: it can be thought of as the fluid mass that goes along with a body when it is accelerated or decelerated. Forces due to added mass will be seen most clearly in experiments under conditions when the body has a low instantaneous speed, and separation drag forces are minimal. The added mass of various two-dimensional sections and three-dimensional shapes can be looked up in tables. As one

simple example, the added mass of a long cylinder exposed to crossflow is precisely the mass of the displaced water: $A_m = \pi r^2 \rho$ (per unit length).

A very interesting and important aspect of added mass is its connection with the Archimedes force. We observe that the added mass force on a body accelerating in a still fluid is only one-half that which is seen on a stationary body in an accelerating flow. Why is this? In the case of the accelerating fluid, and regardless of the body shape or size, there must be a pressure gradient in the direction of the acceleration - otherwise the fluid would not accelerate. This non-uniform pressure field integrated over the body will lead to a force. This is entirely equivalent to the Archimedes explanation of why, in a gravitational field, objects float in a fluid. This effect is not at all present if the body is accelerating in a fluid having no pressure gradient. The "mass" that explains this Archimedes force as an inertial effect is in fact the same as the added mass, and hence the factor of two.

For the development of a simple model, we will focus on a body moving in the vertical direction; we term the vertical motion $\xi(t)$, and it is centered at $x = 0$. The vertical wave elevation is $\eta(t, x)$, and the vertical wave velocity is $w(t, x, z)$. The body has beam $2b$ and draft T; its added mass in the vertical direction is taken as A_m. The objective is to write an equation of the form

$$m\xi_t t + C\xi = F_I + F_D + F_R,$$

where m is the material (sectional) mass of the vessel, and C is the hydrostatic stiffness, the product of ρg and the waterplane area: $C = 2b\rho g$.

The diffraction force is

$$F_D(t) = A_m w_t(t, x = 0, z = -T/2).$$

In words, this force pushes the body upward when the wave is accelerating upward. Note the wave velocity is referenced at the center of the body. This is the effect of the accelerating flow encountering a fixed body - but does *not* include the Archimedes force. The Archimedes force is derived from the dynamic pressure in the fluid independent of the body, and captured in the incident wave force below. The radiation force is

$$F_R(t) = -A_m \xi_{tt}.$$

This force pulls the body downward when it is accelerating upward; it is the effect of the body accelerating through still fluid. Clearly there is no net force when the acceleration of the wave is matched by the acceleration of the body: $F_D + F_R = 0$.

Now we describe the incident wave force using the available descriptions from the linear wave theory:

$$\begin{aligned} \eta(t, x) &= a\cos(\omega t - kx + \psi) \text{ and} \\ p(t, x, z) &= \rho g a e^{kz}\cos(\omega t - kx + \psi) - \rho g z. \end{aligned}$$

We will neglect the random angle ψ and the hydrostatic pressure $-\rho gz$ in our discussion. The task is to integrate the pressure force on the bottom of the structure:

$$
\begin{aligned}
F_I &= \int_{-b}^{b} p(t, x, z = -T)dx \\
&= \rho a g e^{-kT} \int_{-b}^{b} \cos(\omega t - kx)dx \\
&= \frac{2\rho a g}{k} e^{-kT} \cos(\omega t) \sin(kb).
\end{aligned}
$$

As expected, the force varies as $\cos(\omega t)$. The effect of spatial variation in the x-direction is captured in the $\sin(kb)$ term.

If $kb < 0.6$ or so, then $\sin(kb) \approx kb$. This is the case that b is about one-tenth of the wavelength or less, and quite common for smaller vessels in beam seas. Further, $e^{-kT} \approx 1 - kT$ if $kT < 0.3$ or so. This is true if the draft is less than about one twentieth of the wavelength, also quite common. Under these conditions, we can rewrite F_I:

$$
\begin{aligned}
F_I &\approx 2\rho g a (1 - kT) b \cos \omega t \\
&= 2b\rho g a \cos \omega t - 2bT\rho \omega^2 a \cos \omega t \\
&= C\eta(t, x = 0) + \nabla \rho w_t(t, x = 0, z = 0).
\end{aligned}
$$

Here ∇ is the (sectional) volume of the vessel. Note that to obtain the second line we used the deepwater dispersion relation $\omega^2 = kg$.

We can now assemble the complete equation of motion:

$$
\begin{aligned}
m\xi_{tt} + C\xi &= F_I + F_D + F_R \\
&= C\eta(t, x = 0) + \nabla \rho w_t(t, x = 0, z = 0) + A_m w_t(t, x = 0, z = -T/2) - \\
&\quad A_m \xi_{tt}, \text{ so that} \\
(m + A_m)\xi_{tt} + C\xi &\approx C\eta(t, x = 0) + (\nabla \rho + A_m)w_t(t, x = 0, z = -T/2).
\end{aligned}
$$

Note that in the last line we have equated the z-locations at which the fluid acceleration w_t is taken, to $z = -T/2$. It may seem arbitrary at first, but if we chose the alternative of $w_t(t, x = 0, z = 0)$, we would obtain

$$
\begin{aligned}
(m + A_m)\xi_{tt} + C\xi &= C\eta(t, x = 0) + (\nabla \rho + A_m)w_t(t, x = 0, z = 0) \\
(-(m + A_m)\omega^2 + C)\xi &= (C - (\nabla \rho + A_m)\omega^2)\eta(t, x = 0) \longrightarrow \\
\frac{\xi(j\omega)}{\eta(j\omega)} &= 1,
\end{aligned}
$$

since m is equal to $\nabla \rho$ for a neutrally buoyant body. Clearly the transfer function relating vehicle heave motion to wave elevation cannot be unity - the vessel does not follow all waves

equally! If we say that $w_t(t, x = 0, z = -T/2) = \gamma w_t(t, x = 0, z = 0)$, where $\gamma < 1$ is a function of the wavelength and T, the above becomes more suitable:

$$
\begin{aligned}
(-(m + A_m)\omega^2 + C)\xi &= (C - (\gamma \nabla \rho + A_m)\omega^2)\eta(t, x = 0) \longrightarrow \\
\frac{\xi(j\omega)}{\eta(j\omega)} &= \frac{C - (\gamma \nabla \rho + A_m)\omega^2}{C - (m + A_m)\omega^2}
\end{aligned}
$$

This transfer function has unity gain at low frequencies and gain $(\gamma \nabla \rho + A_m)/(m + A_m)$ at high frequencies. It has zero magnitude at $\omega = \sqrt{C/(\gamma \nabla \rho + A_m)}$, but very high magnitude (resonance) at $\omega = \sqrt{C/(m + A_m)}$. The zero occurs at a higher frequency than the resonance because $\gamma < 1$.

In practice, the approximation that w_t should be taken at $z = -T/2$ is reasonable. However, one significant factor missing from our analysis is damping, which depends strongly on the specific shape of the hull. Bilge keels and sharp corners cause damping, as does the creation of radiated waves.

6.7   Limits of the Linear Theory

The approximations made inevitably affect the accuracy of the linear wave model. Here are some considerations. The ratio of wave height to wavelength is typically in the range of 0.02-0.05; a one-meter wave in sea state 3 has wavelength on the order of fifty meters. When this ratio approaches 1/7, the wave is likely to break. Needless to say, at this point the wave is becoming nonlinear! Ultimately, however, even smaller ocean waves interact with each other in a nonlinear way. There are effects of bottom topography, wind, and currents. Nonlinear interaction causes grouping (e.g., rogue waves), and affects the propagation and directionality of waves. It is impossible to make a forward prediction in time - even with perfect and dense measurements - of a wave field, unless these effects are included. In fact, there is some recent work by Yue's group at MIT on the large-scale prediction problem.

6.8   Characteristics of Real Ocean Waves

The origin of almost all ocean waves is wind. Tides and tsunamis also count as waves, but of course at different frequencies. Sustained wind builds waves bigger in amplitude and longer in wavelength - hence their frequency decreases. Waves need sufficient physical space, called the fetch, to fully develop. When the wind stops (or the wave moves out of a windy area), the amplitude slowly decays, with characteristic time $\tau = g^2/2\nu\omega^4$. This rule says that low-frequency waves last for a very long time!

The spectra of ocean waves are reasonably modeled by the standard forms, including JON-SWAP, Pierson-Moskowitz, Ochi, and Bretschneider; these have different assumptions and different applications. The conditions of building seas and decaying seas (swell) are different; in the former case, the spectrum is quite wide whereas it may be narrow for the latter.

Further details can be found in subject texts, including the Principles of Naval Architecture (E.V., Lewis, ed. SNAME, 1989).
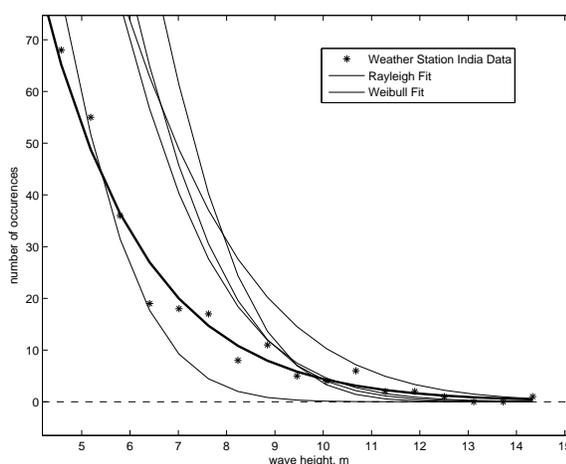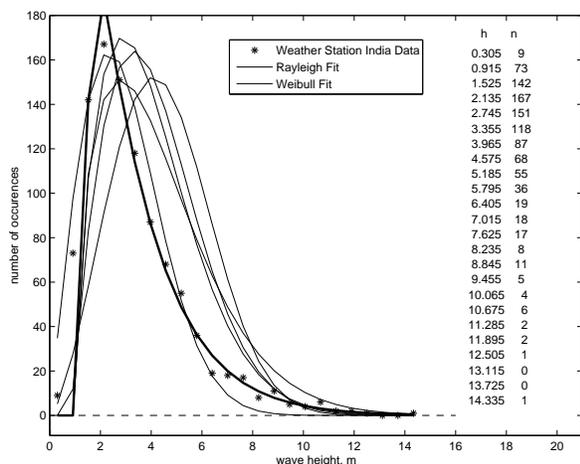
Most important from a design point of view, it has been observed that extreme events do NOT follow the Rayleigh distribution - they are more common. Such dangers are well documented in data on a broad variety of processes including weather, ocean waves, and some social systems. In the case of ocean waves, nonlinear effects play a prominent role, but a second factor which has to be considered for long-term calculations is storms. In periods of many years, intense storms are increasingly likely to occur, and these create short-term extreme seas that may not be well characterized at all in the sense of a spectrum. For the purpose of describing such processes, the Weibull distribution affords some freedom in shaping the "tail." The Weibull cpf and pdf are respectively:

$$
\begin{aligned}
P(h < h_o) &= 1 - e^{-(x-\mu)^c/b^c}; \\
p(h) &= \frac{c(x-\mu)^{c-1}}{b^c} e^{-(x-\mu)^c/b^c}.
\end{aligned}
$$

It is the choice of c to be a (real) number other than two which makes the Weibull a more general case of the Rayleigh distribution. b is a measure related to the standard deviation, and μ is an offset applied to the argument, giving further flexibility in shaping. Clearly $x > \mu$ is required if c is non-integer, and so μ takes the role of a lower limit to the argument. No observations of h below μ are accounted for in this description.

Here is a brief example to illustrate. Data from Weather Station India was published in 1964 and 1967 (see *Principles of Naval Architecture*), giving a list of observed wave heights taken over a long period. The significant wave height in the long-term record is about five meters, and the average period is about ten seconds. But the distribution is decidedly non-Rayleigh, as shown in the right figure below. Several trial Weibull pdf's are shown, along with an optimal (weighted least-squares) fit in the bold line. The right figure is a zoom of the left, in the tail region.

h	n
0.305	9
0.915	73
1.525	142
2.135	167
2.745	151
3.355	118
3.965	87
4.575	68
5.185	55
5.795	36
6.405	19
7.015	18
7.625	17
8.235	8
8.845	11
9.455	5
10.065	4
10.675	6
11.285	2
11.895	2
12.505	1
13.115	0
13.725	0
14.335	1

Armed with this distribution, we can make the calculation from the cpf that the 100-year wave is approximately 37 meters, or $7.5\bar{h}^{1/3}$. This is a very significant amplification, compared to the factor of three predicted using short-term statistics in Section 5.6, and reinforces the importance of observing and modeling accurately real extreme events.

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