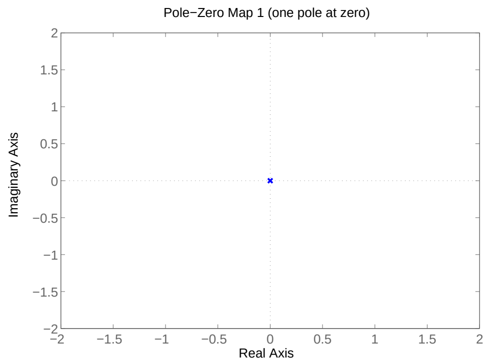
MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Department of Mechanical Engineering

2.004 Dynamics and Control II
Fall 2007

Lecture 38
Overview of the frequency response and Bode plots

In this lecture, we'll practice on the topic of frequency response and Bode plots. Select at least one partner, and work out the problems together. Call on the instructor(s) for help.

1. Consider the following eight distinct systems whose pole–zero plots are provided. Match them their bode plots given in the next page.



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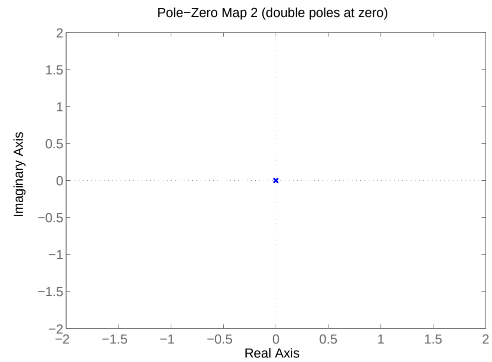

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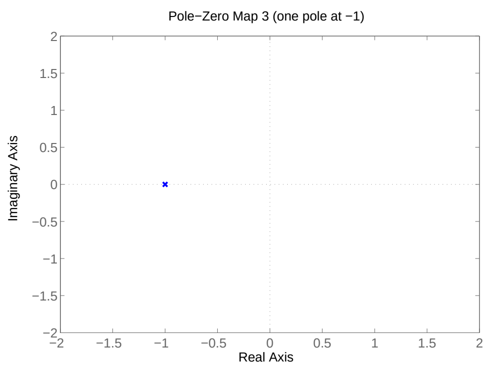

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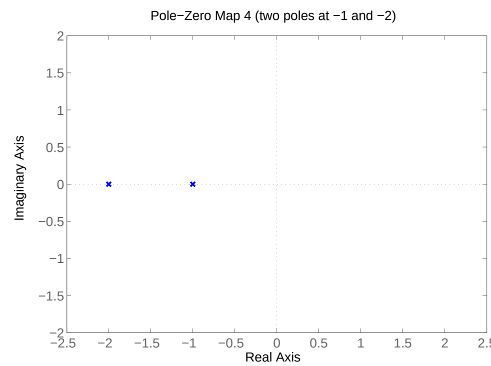

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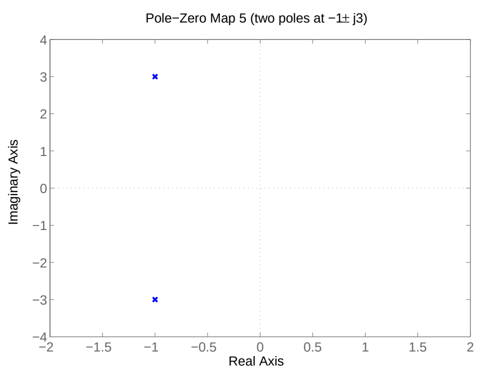

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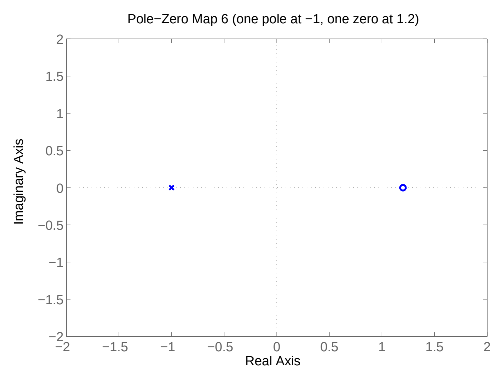

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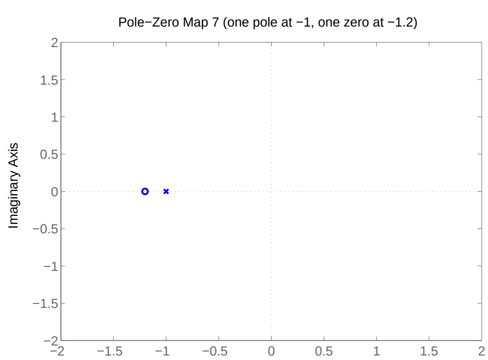

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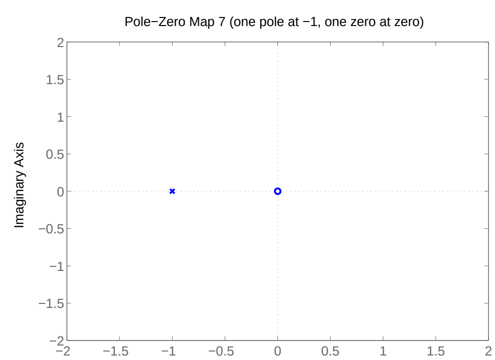

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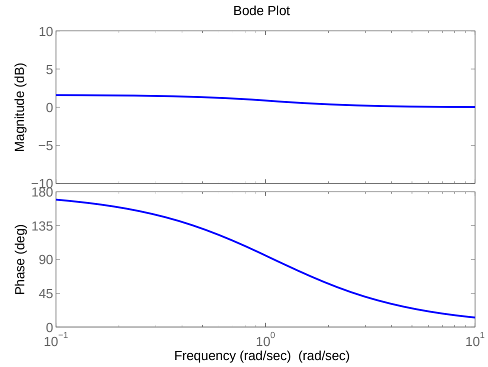
Bode plots:



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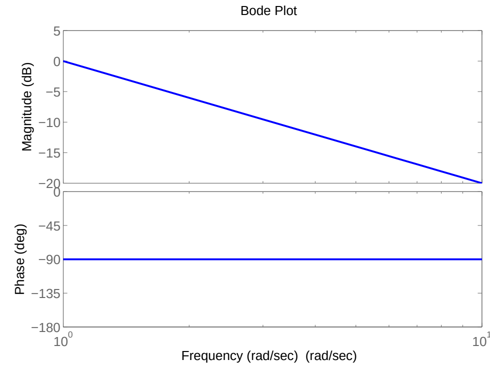

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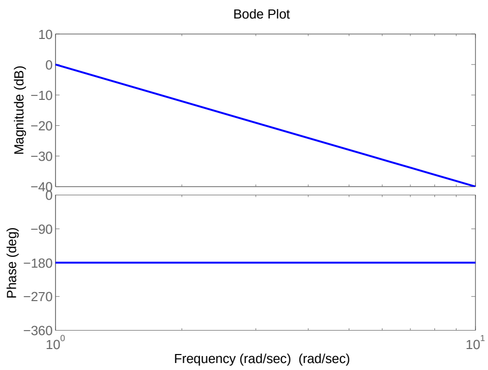

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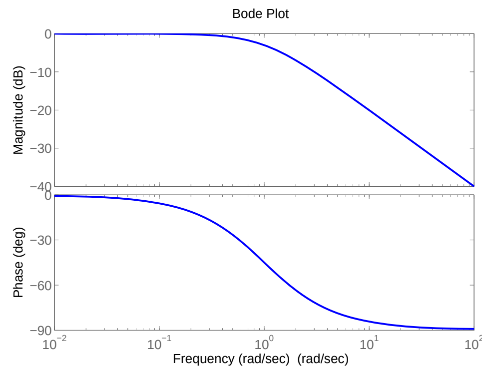

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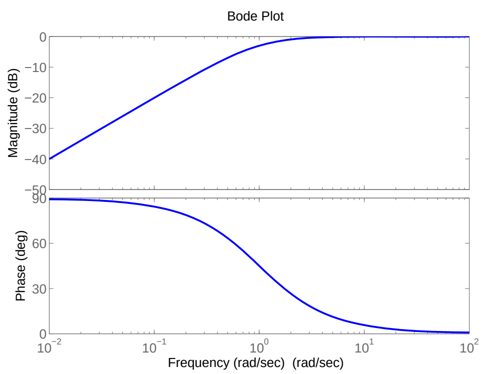

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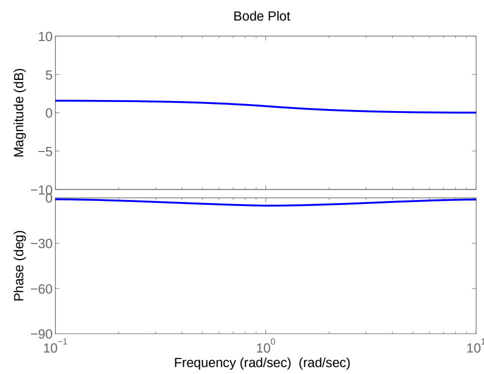

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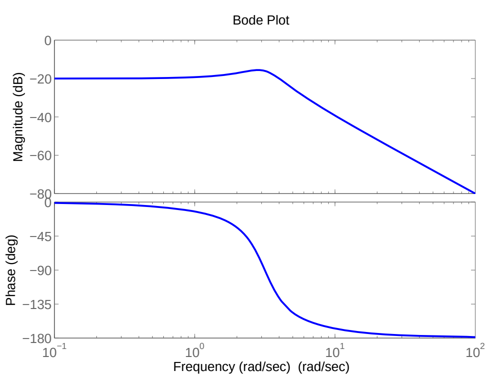

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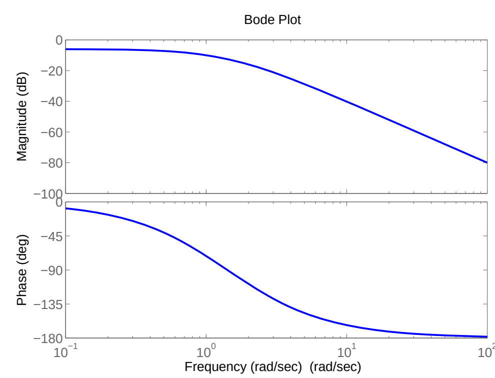

h

2. Nise chapter 10, problem 25 (page 679).

3. Nise chapter 10, problem 23 (page 679).

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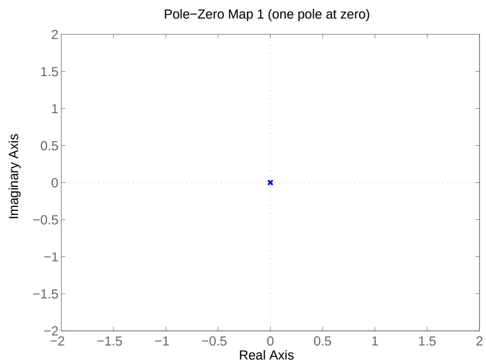
Supplement to Lecture 38
Solution to the practice problems on frequency response & Bode plots

1. Consider the following eight distinct systems whose pole–zero plots are provided. Match them their bode plots given in the next page.

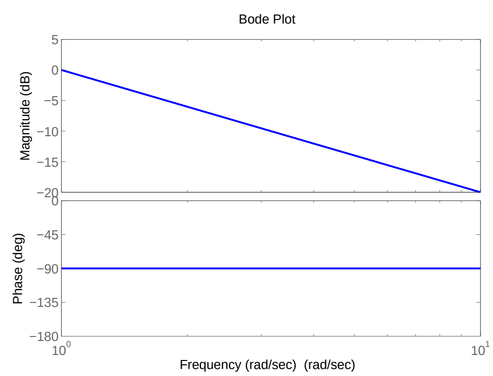
1.1 *Answer:*



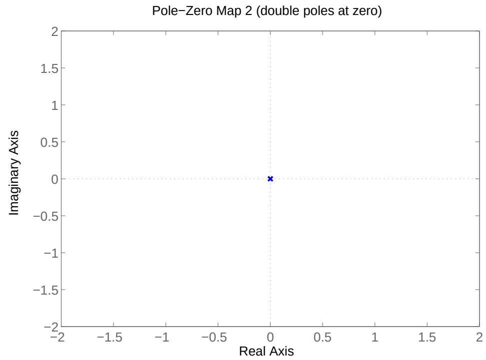
This system has a pole at $s = 0$, its transfer function will be

$$G_a(s) = \frac{K}{s},$$

where K is a gain. The Bode magnitude plot has -20dB/dec slope and the Bode phase plot is always $-90°$.



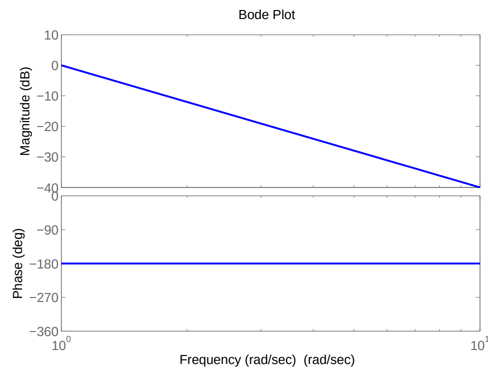
1.2 *Answer:*



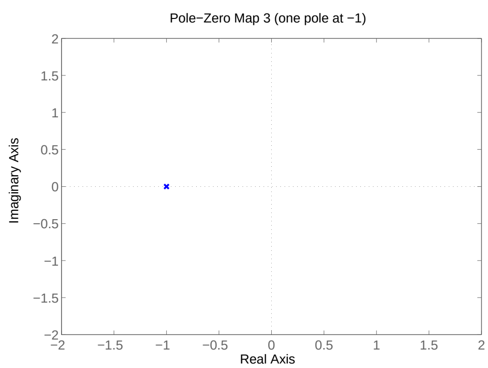
Because this system has double poles at $s = 0$, the transfer function will be

$$G_b(s) = \frac{K}{s^2}.$$

The Bode magnitude plot has -40dB/dec slope and the Bode phase plot is always $-180°$.



1.3 *Answer:*

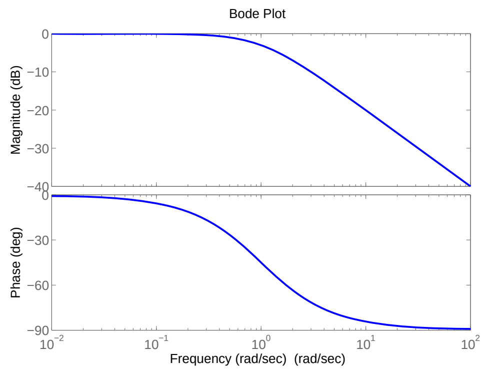


This system has a pole at $s = -1$, the transfer function will be

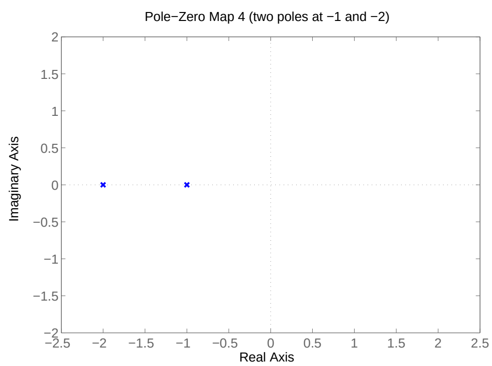
$$G_3(s) = \frac{K}{s+1}.$$

The break frequency is $\omega = 1\,(\text{rad/s})$. Before the break frequency, the Bode magnitude plot is flat and after the break frequency it decreases with -20

dB/dec slope. In the Bode phase plot, the phase at low frequencies is $0°$ and decreases to $-90°$ over 2 decade bandwidth around the break frequency.



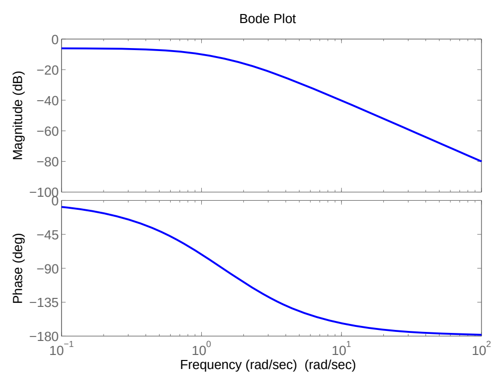
1.4 *Answer:*



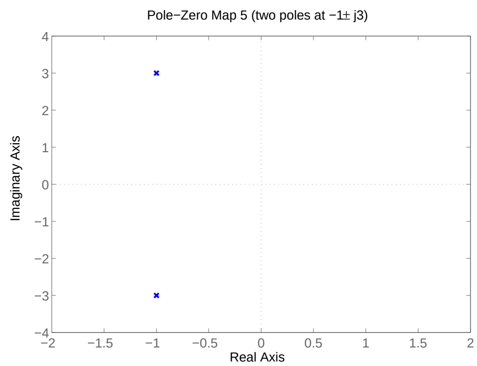
This system has two poles at $s = -1$ and $s = -2$. The transfer function will be

$$G_4(s) = \frac{K}{(s+1)(s+2)}.$$

The break frequencies are $\omega = 1$ (rad/s) and $\omega = 2$ (rad/s). The Bode magnitude plots indicates -20dB/dec and the Bode phase plots indicates $-90°$ at each break frequency.



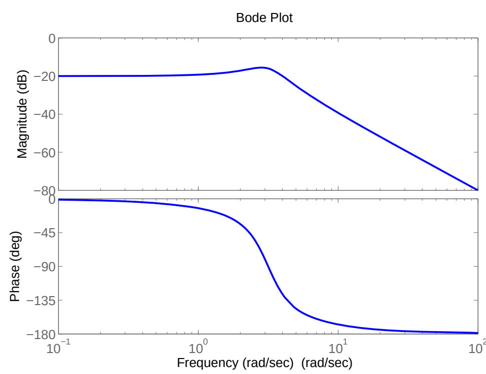
1.5 *Answer:*



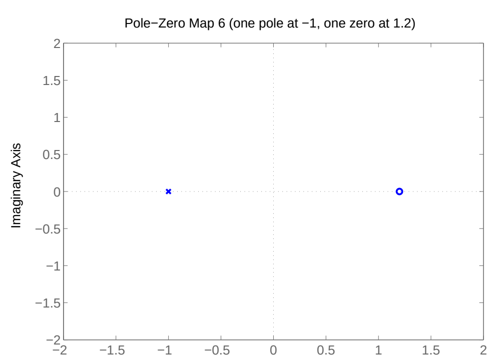
This system has a pair of complex poles at $s = -1 \pm j3$. The transfer function will be

$$G_5(s) = \frac{K}{(s-(-1+j3))(s-(-1-j3))} = \frac{K}{s^2+2s+10},$$

where $\omega_n = \sqrt{10}$ and $\zeta = 1/\sqrt{10}$. Since $\zeta < 1/\sqrt{2}$, the resonance peak should be observable. The peak frequency is $\omega_p = \omega_n\sqrt{1-2\zeta^2} = 2.82(\text{rad/s})$ and the correction is $-20\log(2\zeta) \approx 4$ dB. In Bode phase plot, at low frequencies, the phase is $0°$ and it decreases to $-180°$ over 2 decades bandwidth around the peak frequency.



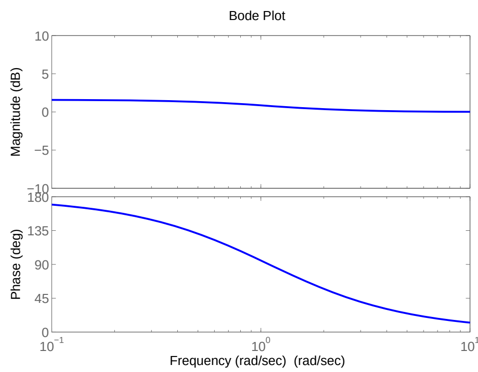
1.6 *Answer:*



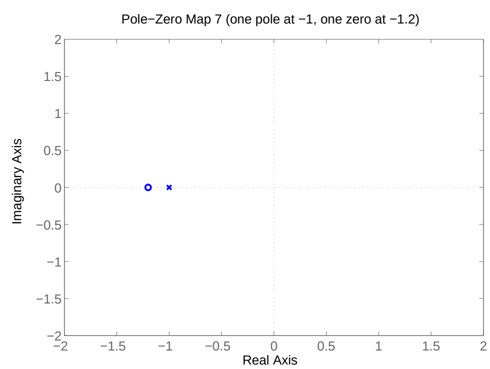
This system has a pole at $s = -1$ and a zero at $s = 1.2$. The transfer function will be

$$G_6(s) = \frac{K(s-1.2)}{(s+1)}.$$

The Bode magnitude plot gets +20 dB/dec at the zero and -20 dB/dec at the pole. Therefore, at the low and high frequencies, the Bode magnitude plot would be flat. Since the two break frequencies are very close, the Bode magnitude plot looks almost flat. In the Bode phase plot, the zero in the right–hand plane has the same effects as the pole in the left–hand plane, the total phase change is $-180°$. When you plug in $s = 0$, then $g_6(s = 0) = -K/1.2$, which is a negative value. Thus the Bode phase plot starts at $-180°$ or $180°$ and decreases by $-180°$.



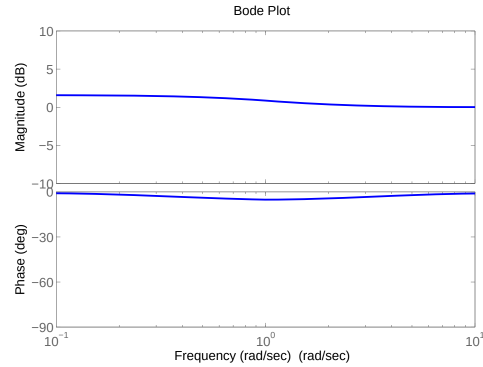
1.7 *Answer:*



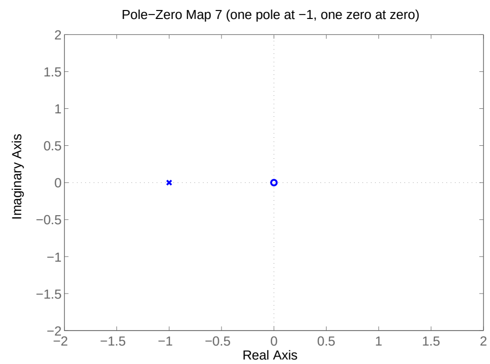
This system has a pole $s = -1$ and a zero at $s = -1.2$. The transfer function will be

$$G_7(s) = \frac{K}{(s+1)(s+1.2)}.$$

The Bode magnitude plot is identical as the case of (g) since they have the same break frequencies. Because the zero is on the left–hand plane, the total phase is zero.



1.8 *Answer:*



The system has a pole at $s = -1$ and a zero at $s = 0$. The transfer function will be

$$G_8(s) = \frac{Ks}{s+1}.$$

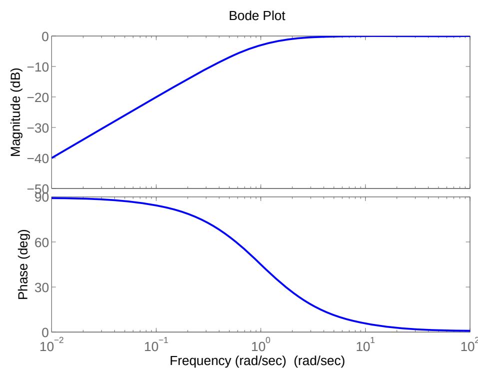
Because of the zero at $s = 0$, the Bode magnitude plot starts with 20dB/dec slope and it becomes flat after the break frequency at $\omega = 1\,(\text{rad/s})$. To find the Bode phase plot, let's compute the phase analytically.

$$G_8(j\omega) = K\frac{j\omega}{1+j\omega} = \frac{j\omega(1-j\omega)}{(1+j\omega)(1-j\omega)} = \frac{\omega^2 + j\omega}{1+\omega^2}.$$

Thus the phase is

$$\tan^{-1}\left(\frac{1}{\omega}\right).$$

If ω is very small, then $1/\omega$ goes to the infinity, which the phase is 90°. After the break frequency at $\omega = 1$ (rad/s), the phase decreases by -90° due to the pole.



2. Nise chapter 10, problem 25 (page 679).

a. Find the gain margin, phase margin, zero dB frequency, 180° frequency, and the closed–loop bandwidth.

Answer: From the given Bode plot, the zero dB frequency is $\omega \approx 2$ (rad/s) and the phase at the zero dB frequency is $-130°$. Therefore the phase margin is $-130 - (-180) = 50°$. The 180° frequency is about 6 (rad/s), whose gain is about -12dB. Thus the gain margin is 12dB.

To find the closed–loop bandwidth, we find the frequency at which the open–loop magnitude response is between -6 and -7.5 dB if the open–loop phase response is between $-135°$ and $-225°$. From the given bode plot, the frequency is approximately 3 (rad/s), thus $\omega_{BW} = 3$ (rad/s). (Refer to page 653 of Nise)

b. Use your result in (a) to estimate the damping ratio, percent overshoot, settling time, and peak time.

Answer: From the figure 10.48 in page 653 of Nise (or page 8 of Lecture note 32), $\zeta \approx 0.45$.

$$\%\text{OS} = \exp\left\{-\frac{\zeta\pi}{\sqrt{1-\zeta^2}}\right\} = 20.5\%.$$

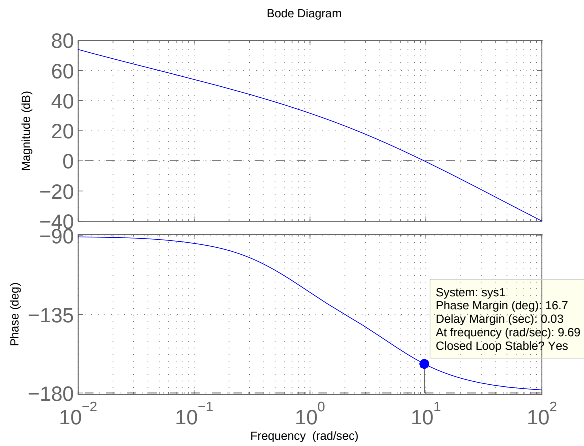
From the figure 10.41 in page 644 of Nise (or page 7 of Lecture note 32),

$$T_s = \frac{4}{\omega_{WB}\zeta}\sqrt{(1-2\zeta^2)+\sqrt{4\zeta^4-4\zeta^2+2}} \approx 3.92 \text{ (s)s}.$$

$$T_p = \frac{\pi}{\omega_{WB}\sqrt{1-\zeta^2}}\sqrt{(1-2\zeta^2)+\sqrt{4\zeta^4-4\zeta^2+2}} \approx 1.551 \text{ (s)}.$$

3. Nise chapter 10, problem 23 (page 679).

a. System 1 *Answer:* From the following Bode plot,



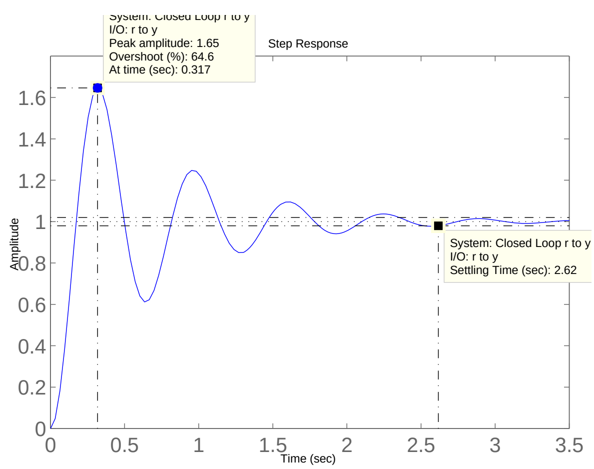
the phase margin is 16.7°. The damping ratio ζ is approximately 0.15. Using the same method in question (2), the closed–loop bandwidth $\omega_{WB} = 14$ (rad/s). Thus

$$\%\mathrm{OS} = \exp\left\{-\frac{\zeta\pi}{\sqrt{1-\zeta^2}}\right\} \approx 62\%.$$

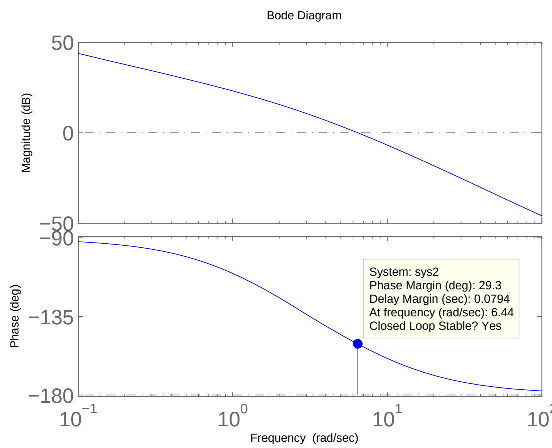
$$T_s = \frac{4}{\omega_{WB}\zeta}\sqrt{(1-2\zeta^2)+\sqrt{4\zeta^4-4\zeta^2+2}} \approx 2.9\ (\mathrm{s})\mathrm{s}.$$

$$T_p = \frac{\pi}{\omega_{WB}\sqrt{1-\zeta^2}}\sqrt{(1-2\zeta^2)+\sqrt{4\zeta^4-4\zeta^2+2}} \approx 0.35\ (\mathrm{s}).$$

Note that it approximately agrees with the exact step response.



b. System 2 *Answer:* From the following Bode plot,



the phase margin is 29.3°. The damping ratio ζ is 0.25. Using the same method in question (2), the closed–loop bandwidth $\omega_{WB} = 10\,(\text{rad/s})$. Thus

$$\%\mathrm{OS} = \exp\left\{-\frac{\zeta\pi}{\sqrt{1-\zeta^2}}\right\} \approx 44\%.$$

$$T_s = \frac{4}{\omega_{WB}\zeta}\sqrt{(1-2\zeta^2)+\sqrt{4\zeta^4-4\zeta^2+2}} \approx 2.4\,(\text{s})\text{s}.$$

$$T_p = \frac{\pi}{\omega_{WB}\sqrt{1-\zeta^2}}\sqrt{(1-2\zeta^2)+\sqrt{4\zeta^4-4\zeta^2+2}} \approx 0.48\,(\text{s}).$$

Note that it approximately agrees with the exact step response.