2.003 Engineering Dynamics

Problem Set 11

Problem 1: Torsional Oscillator

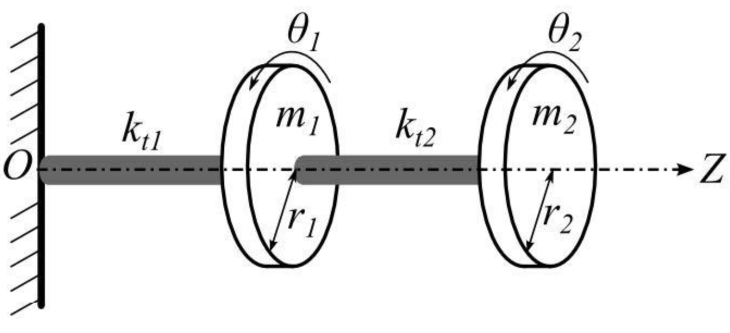

Figure 1: Torsional oscillator

Two disks of radius r_1 and r_2 and mass m_1 and m_2 are mounted in series with steel shafts. The shaft between the base and m_1 has length L_1 and the shaft from m_1 to m_2 has length L_2. The shafts have cross section polar moments of inertia, J_{zz1} and J_{zz2}. They also have shear moduli, G_1 and G_2. The torsional spring constant of each shaft is given by $k_{t1} = \frac{G_1 J_1}{L_1}$ and $k_{t2} = \frac{G_2 J_2}{L_2}$. the mass moments of inertia of the disks are given by $I_{zz1} = \frac{1}{2} m_1 r_1^2$ and $I_{zz2} = \frac{1}{2} m_2 r_2^2$. The disks rotate about the OZ axis. The physical properties are such that $\frac{I_{zz1}}{I_{zz2}} = \frac{k_{t1}}{k_{t2}} = 10$ and $I_{zz1} = 0.2\, kg - m^2$, and $k_{t1} = 4000\, N - m / radian.$

a) Find the equation of motion of the system and write it in matrix form.

b) Find the undamped natural frequencies of the system ω_1 and ω_2.

b) Find the undamped mode shapes of the system.

Concept question: If the two rotors are given initial torsional deflections, which are exactly in proportion to the mode shape of mode 2, what frequency components do you expect to see in the resulting transient response of the system. (a) ω_{n1} only, , (b) ω_{n2} only, (c) ω_{n1} and ω_{n2}.

Concept question answer: (b) is correct. If the initial condition is in the shape of one mode, only that mode participates in the response, and it does so at its natural frequency.

Problem 2: 2 DOF cart with a slender rod

This exact system was examined in problem set 10. A slender rod of length *l* (2m) and mass m_2(0.5kg) is attached to a pivot at A to a block of mass m_1(1kg). In problem set 10 the pivot was assumed to be frictionless. In this problem the resistance at the pivot is modeled as a linear torsional damper with torsional damping coefficient c =0.025 N-m-s/rad. The block moves horizontally on rollers. The position of the block, relative to the un-stretched spring position is described by the x coordinate. The block is connected to a fixed wall by a spring of constant k=10 N/mwith an un-stretched length l_o(0.5m),and a linear damper with damping coefficient b =0.05N-s/m.

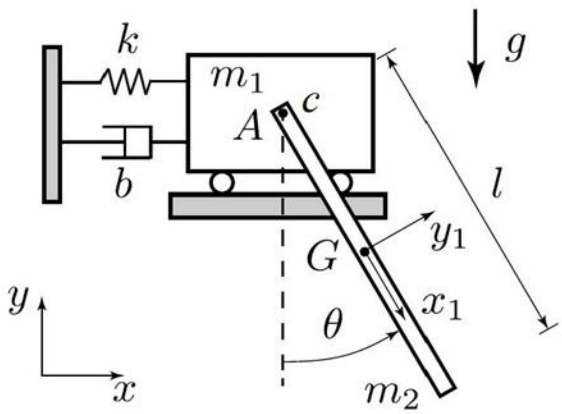

Figure 2: Cart with a slender rod

The linearized equations of motion for the system are the same as found in problem set 10 except that an external torque, $-c\dot{\theta}$, resulting from the linear damper at the pivot must be included in Equation 2 below, the rotational equation of motion:

$$\left(m_1+m_2\right)\ddot{x}+b\dot{x}+kx+m_2\frac{l}{2}\left(\ddot{\theta}\cos\theta-\dot{\theta}^2\sin\theta\right)=0 \tag{1}$$

$$m_2\ddot{x}\frac{l}{2}\cos\theta+m_2\frac{l^2}{3}\ddot{\theta}+m_2 g\frac{l}{2}\sin\theta+c\dot{\theta}=0,\text{ and} \tag{2}$$

In the previous problem set the linearized equations were found. When the additional torsional damping term is included the linearized equations become:

$$\left(m_1+m_2\right)\ddot{x}+b\dot{x}+kx+m_2\frac{l}{2}\ddot{\theta}=0, \text{ and} \quad (3)$$

$$m_2\ddot{x}\frac{l}{2}+m_2\frac{l^2}{3}\ddot{\theta}+c\dot{\vartheta}+m_2 g\frac{l}{2}\theta=0,$$

These may be expressed in matrix form:

$[M]\begin{Bmatrix}\ddot{x}\\ \ddot{\theta}\end{Bmatrix}+[C]\begin{Bmatrix}\dot{x}\\ \dot{\theta}\end{Bmatrix}+[K]\begin{Bmatrix}x\\ \theta\end{Bmatrix}=0,$ where [M], [C] and [K] are known as the mass, damping and stiffness matrices.

$$\begin{bmatrix}(m_1+m_2) & m_2\frac{l}{2}\\ m_2\frac{l}{2} & m_2\frac{l^2}{3}\end{bmatrix}\begin{Bmatrix}\ddot{x}\\ \ddot{\theta}\end{Bmatrix}+\begin{bmatrix}b & 0\\ 0 & c\end{bmatrix}\begin{Bmatrix}\dot{x}\\ \dot{\theta}\end{Bmatrix}+\begin{bmatrix}k & 0\\ 0 & m_2 g\frac{l}{2}\end{bmatrix}\begin{Bmatrix}x\\ \theta\end{Bmatrix}=\begin{Bmatrix}0\\ 0\end{Bmatrix} \quad (4)$$

Substitution of the physical quantities in these equations of motion yields:

$$\begin{bmatrix}1.5kg & 0.5kg-m\\ 0.5kg-m & 0.667kg-m^2\end{bmatrix}\begin{Bmatrix}\ddot{x}\\ \ddot{\boldsymbol{\theta}}\end{Bmatrix}+\begin{bmatrix}0.05N-s/m & 0\\ 0 & 0.025N-m-s\end{bmatrix}\begin{Bmatrix}\dot{x}\\ \dot{\boldsymbol{\theta}}\end{Bmatrix}+\begin{bmatrix}10N/m & 0\\ 0 & 4.905N-m\end{bmatrix}\begin{Bmatrix}x\\ \boldsymbol{\theta}\end{Bmatrix}=0$$

The units are shown to emphasize that a consistent set must be used. Without showing the units, a very compact statement of the equations of motion results:

$$\begin{bmatrix}1.5 & 0.5\\ 0.5 & 0.667\end{bmatrix}\begin{Bmatrix}\ddot{x}\\ \ddot{\boldsymbol{\theta}}\end{Bmatrix}+\begin{bmatrix}0.05 & 0.0\\ 0.0 & 0.025\end{bmatrix}\begin{Bmatrix}\dot{x}\\ \dot{\boldsymbol{\theta}}\end{Bmatrix}+\begin{bmatrix}10 & 0\\ 0 & 4.905\end{bmatrix}\begin{Bmatrix}x\\ \boldsymbol{\theta}\end{Bmatrix}=0. \quad (5)$$

In Problem Set #10 the undamped, linearized equations of motion were used to find the following undamped natural frequencies and mode shapes.

$\boldsymbol{\omega}_1=2.158$ radians/second

$\boldsymbol{\omega}_2=3.746$ radians/second

Mode shapes:

Mode 1: $\begin{Bmatrix}X/X\\ \boldsymbol{\Theta}/X\end{Bmatrix}^{(1)}=\begin{Bmatrix}1.0\\ 1.295\end{Bmatrix}$

Mode 2: $\begin{Bmatrix}X/X\\ \boldsymbol{\Theta}/X\end{Bmatrix}^{(2)}=\begin{Bmatrix}1.0\\ -1.5749\end{Bmatrix}$

These mode shapes have been normalized by dividing both elements of the mode shape by the top element,x. This forces the mode shape to be normalized such that the top element is 1.0. This information is used in this problem to guide the student through a step by step application of the method of modal analysis. In modal analysis the total system response is found in terms of each modes contribution to the total. The first step is to define a matrix of mode shapes [U].

a) Express the mode shapes, given above, as a matrix in the following form:

$$[U]=\left[\{u\}^{(1)}\{u\}^{(2)}\right]=\left[\begin{Bmatrix}1\\ u_2/u_1\end{Bmatrix}^{(1)}\begin{Bmatrix}1\\ u_2/u_1\end{Bmatrix}^{(2)}\right].$$

[U] is the matrix of mode shape vectors, which in this case has been normalized so that the top element of each vector is equal to 1. Find U^T the transpose of U, and find the inverse of U. The last is most easily done numerically with a program such as Matlab. But for a 2x2 matrix it may also be done directly by hand computation. Refer to any linear algebra text.

b) Compute $U^T MU,\ U^T KU,$ and $U^T CU,$ the modal mass, modal stiffness and modal damping matrices. These matrices should be diagonal to within the accuracy of round off errors. Are they? If not you may not have carried enough significant digits when entering the mode shapes in carrying out the computations. Considerable precision is required.

c) The system is given an initial displacement $x_o = 0.0,\ \theta_o = 0.2 radians.$ Find an expression for the transient response of $\theta(t)$.

d) A horizontal force $F(t) = F_o \cos(\omega_{n2} t)$ is now applied to the cart. Find the steady state response, $\theta(t)$. Note that the excitation frequency is at one of the natural frequencies of the system.

Concept Question: Applying the concept of modal analysis, which mode is likely to dominate the steady state response for the excitation prescribed in part d?

(a) mode 1, (b) mode 2, (c) Neither

Concept question answer: (b) is correct. The frequency of excitation is at the natural frequency of mode 2. With light damping the mode with resonant response will dominate.

Problem 3: Pump with an imbalanced rotor

For a system similar to that in Problem 2, a servo motor, mounted as a part of the cart, drives the rod at a constant rotation rate Ω, such that $\theta(t) = \Omega t$. In this case, Let the length of the rotating rod be l =0.02 m and m_2 =0.1 kg. The other values are unchanged. This system may now be represented with a single equation of motion, because one coordinate is sufficient to describe the response of the system. That coordinate is the motion of the cart x(t). The rotation of the arm is no longer an unknown response, but a specified input.

This system might be a pump with a statically imbalanced rotor. For the purpose of this problem, assume that the spring and damper connecting the pump to the wall have been added in an attempt to vibration isolate the pump from the wall. This is intended to reduce the dynamic force transmitted to the wall. The rotating mass may be modeled as an external harmonic exciting force in the equation of motion. Let $F_o(t)$ be the equivalent horizontal component of this force.

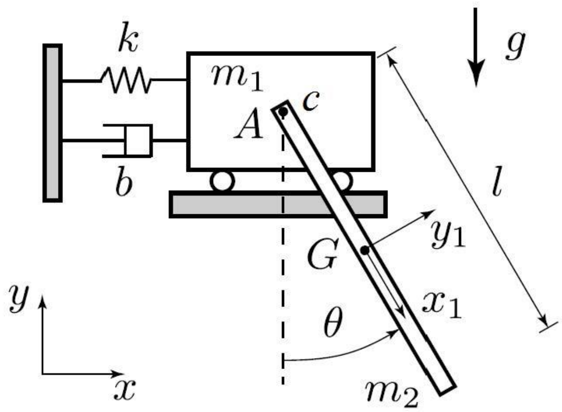

Figure 2: Cart with a slender rod

a) Find and expression for $F_o(t)$, the horizontal component of the force caused by the rotating mass.

b) Compute F_T/F_o, the ratio of the steady state magnitude of the force transmitted to the wall to the magnitude of the exciting force caused by the rotating mass for the following cases: $\frac{\Omega}{\omega_n} = 0.1, 1.0,\ or\ 5.0.$

Concept question: For what value of $\frac{\Omega}{\omega_n}$ is the force transmitted to the wall the greatest?

(a) 0.1, (b)1, (c)5

Concept question answer: (b) is correct. When the damping is much less than critical, at resonance the response will be the greatest and the forces transmitted to the wall will also be greatest.

Problem 4: Cart with a massive spring

A cart of mass M is attached to the wall by a spring of constant k, mass m and un-stretched length *lo*. a) Find the equation of motion of the system accounting for the mass of the spring.

Hint: Compute the kinetic energy of the system and account for the kinetic energy of the spring. Note that the velocity of the spring varies from zero at the point of attachment to the wall to $\dot{x}$ at the point of attachment to the moving

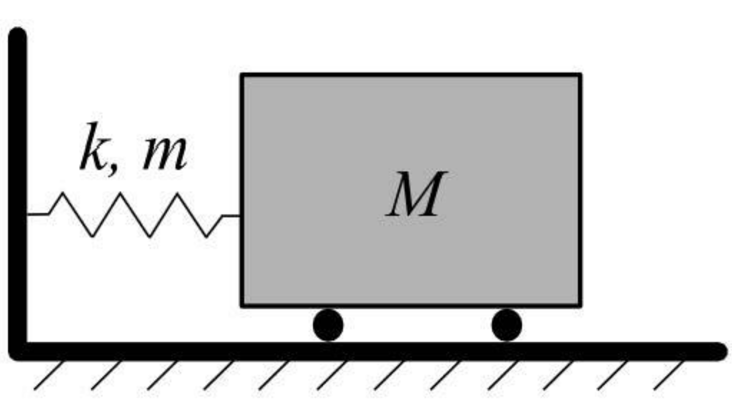

Figure 3: Cart with a massive spring

mass.

b) Compute the ratio $\dfrac{\omega_n}{\omega_{no}}$ where ω_n is the natural frequency of the system accounting for the mass of the spring, and ω_{no} is the natural frequency not including the mass of the spring.

Concept question: Do you expect the mass of the spring to:

(a) increase the computed natural frequency, (b) decrease it, (c) not affect it?

Concept question answer: (b) is correct. Adding inertia to a system lowers the natural frequency.