2.003SC
Recitation 6 Notes: Moment of Inertia and Imbalance, Rotating Slender Rod

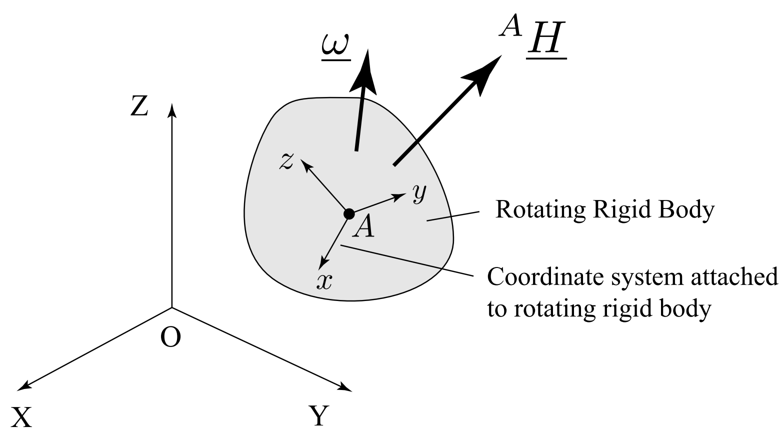
Moment of Inertia

Recall that

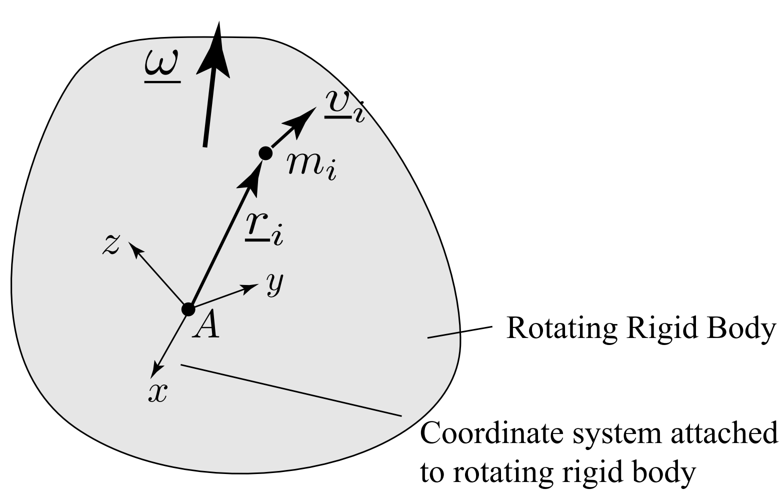
- Frame $Axyz$, the object coordinate system, is attached to (and rotates with) the rotating rigid body
- A is chosen to be EITHER
  fixed in space ($\underline{v}_A = 0$) OR
  the rigid body's center of mass ($\underline{v}_A = \underline{v}_G$)

so

$$\Sigma^A \underline{\tau}_{ext} = \frac{d^A \underline{H}}{dt}$$



If each particle of the rigid body has mass, m_i , a linear velocity, $\underline{v}_i$, and is located by a vector, $\underline{r}_i$,



The angular momentum of the entire rigid body can be determined by summing over all the particles,

$$\underline{H}_A = \Sigma_i(\underline{r}_i \times m_i \underline{v}_i)$$

After some algebra, it can be shown that,

$$[H]_A = [I]_A[\omega]$$

where

$$[I]_A = \begin{bmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{bmatrix} \qquad ; \qquad \underline{\omega} = \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix}$$

The first is the inertia matrix,

$$[I]_A = \begin{bmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{bmatrix} = \begin{bmatrix} \Sigma m_i({y_i}^2 + {z_i}^2) & -\Sigma m_i x_i y_i & -\Sigma m_i x_i z_i \\ -\Sigma m_i y_i x_i & \Sigma m_i({x_i}^2 + {z_i}^2) & -\Sigma m_i y_i z_i \\ -\Sigma m_i z_i x_i & -\Sigma m_i z_i y_i & \Sigma m_i({x_i}^2 + {y_i}^2) \end{bmatrix}$$

Some observations:

- The matrix is symmetric
- The diagonal terms, called the moments of inertia, are always positive
- The off-diagonal terms, called the products of inertia, can be positive or negative

Alignment of $\underline{H}$ and $\underline{\omega}$

The rigid body's angular momentum, $\underline{H}$, and its angular velocity, $\underline{\omega}$, and NOT generally, aligned. Consequently, the angular momentum is changing with time, and since

$$\Sigma {}^A\underline{\tau}_{ext} = \frac{d^A\underline{H}}{dt}$$

external torques must be applied to the body to fix its angular velocity vector in space.

Principal Directions and Principal Moments of Inertia

Recall the choice of object coordinate system, i.e. frame $Axyz$. For every rigid body, there is a set of directions, i.e. a choice of $Axyz$, for which the inertia matrix is diagonal.

$$[I]_A = \begin{bmatrix} I_{xx} & 0 & 0 \\ 0 & I_{yy} & 0 \\ 0 & 0 & I_{zz} \end{bmatrix}$$

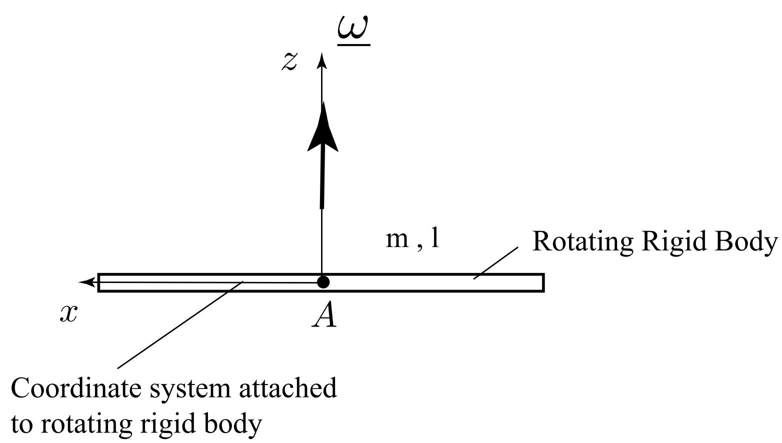
In this situation, the directions are called the principal directions, and the diagonal terms of the inertia matrix are called the principal moments of inertia. The angular momentum expression simplifies to the following.

$$[H]_A = I_{xx}\omega_x + I_{yy}\omega_y + I_{zz}\omega_z$$

The physical significance of the above is that, for a rigid body rotating only about one of its principal axes, the angular momentum vector and the angular velocity vector are aligned, no external torques are required to keep the rotating body in position, and the body is said to be dynamically balanced.

Slendor Rod - Problem Statement 1

A slender rod has a body coordinate system attached such that its x-axis aligns with the rod's long axis. The rod is spinning about its center of mass at an angular velocity, ω , about the z-axis.



Write an expression for the angular momentum, $\underline{H}_A$, and its time derivative, $\frac{d\underline{H}_A}{dt}$.

Slendor Rod - Solution 1

The body coordinate system's axes are the rod's principal directions. The moment of inertia matrix for the rod is as follows.

$$[I] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{12}ml^2 & 0 \\ 0 & 0 & \frac{1}{12}ml^2 \end{bmatrix}$$

The angular velocity of the rod is as follows

$$\underline{\omega} = \begin{bmatrix} 0 \\ 0 \\ \omega \end{bmatrix}$$

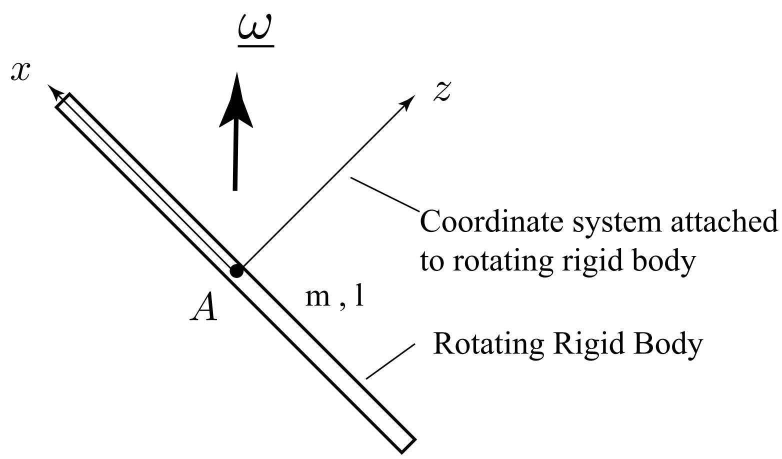
The angular momentum is

$$\underline{H} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{12}ml^2 & 0 \\ 0 & 0 & \frac{1}{12}ml^2 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \omega \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \frac{1}{12}ml^2\omega \end{bmatrix} = \frac{1}{12}ml^2\omega\hat{k}$$

$$\frac{d\underline{H}}{dt} = \frac{1}{12}ml^2\dot{\omega}\hat{k}$$

Slendor Rod - Problem Statement 2

A slender rod has a body coordinate system attached such that its x-axis aligns with the rod's long axis. The rod is spinning about its center of mass at an angular velocity, ω , about an axis between the body coordinate system's x-axis and its z-axis.



Write an expression for the angular momentum, $\underline{H}_A$, and its time derivative, $\frac{d\underline{H}_A}{dt}$.

Slendor Rod - Solution 2

The body coordinate system's axes are the rod's principal directions. The moment of inertia and the angular velocity of the rod are as follows.

$$[I] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{12}ml^2 & 0 \\ 0 & 0 & \frac{1}{12}ml^2 \end{bmatrix} \qquad \underline{\omega} = \begin{bmatrix} \frac{1}{\sqrt{2}}\omega \\ 0 \\ \frac{1}{\sqrt{2}}\omega \end{bmatrix}$$

The angular momentum is

$$\underline{H} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{12}ml^2 & 0 \\ 0 & 0 & \frac{1}{12}ml^2 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}}\omega \\ 0 \\ \frac{1}{\sqrt{2}}\omega \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \frac{1}{12\sqrt{2}}ml^2\omega \end{bmatrix} = \frac{1}{12\sqrt{2}}ml^2\omega\hat{k}$$

$$\frac{d\underline{H}}{dt} = (\frac{1}{12\sqrt{2}}ml^2\dot{\omega})\hat{k} + (\frac{1}{12\sqrt{2}}ml^2\omega)\frac{d\hat{k}}{dt}$$

$$\frac{d\hat{k}}{dt} = \underline{\omega} \times \hat{k} = (\frac{\omega}{\sqrt{2}}\hat{\imath} + \frac{\omega}{\sqrt{2}}\hat{k}) \times \hat{k} = \frac{\omega}{\sqrt{2}}(\hat{\imath} \times \hat{k}) = -\frac{\omega}{\sqrt{2}}\hat{\jmath}$$

$$\Sigma\tau_{ext} = \frac{d\underline{H}}{dt} = (\frac{1}{12\sqrt{2}}ml^2\dot{\omega})\hat{k} - (\frac{1}{24}ml^2\omega^2)\hat{\jmath}$$

The torque vector can be seen to have two components. The first, in the direction of $\hat{k}$, changes the magnitude of H. The second, in the direction of $-\hat{\jmath}$, changes the direction of H.

MIT OpenCourseWare
http://ocw.mit.edu

2.003SC / 1.053J Engineering Dynamics
Fall 2011

For information about citing these materials or our Terms of Use, visit: http://ocw.mit.edu/terms.