2.003J/1.053J Dynamics and Control I, Spring 2007
Professor Thomas Peacock
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Lecture 19

Lagrangian Dynamics: Spinning Hoop with Sliding Mass, Linearization of Equations of Motion, and Bifurcations

Example: Spinning Hoop with Sliding Mass (Continued)

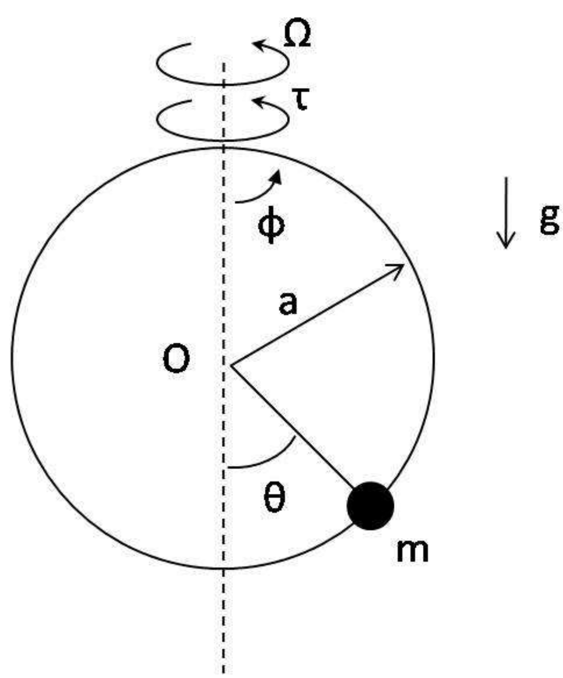

Figure 1: Spinning hoop with sliding mass. Figure by MIT OCW.

Lagrangian

$$L = \frac{1}{2}m(a^2 \sin^2\theta\Omega^2 + a^2\dot{\theta}^2) + (-mga\cos\theta)$$

Lagrange's Equation for θ

$$\ddot{\theta} - \sin\theta\cos\theta\Omega^2 + \frac{g}{a}\sin\theta = 0 \tag{1}$$

Equilibrium Points

$\theta = 0, \pi, \arccos\frac{g}{a\Omega^2}$
The third point only exists if $\frac{g}{a\Omega^2} \leq 1$.

Stability Analysis

Stability around $\theta_e = \arccos(g/a\Omega^2)$

$\theta_e = \arccos\frac{g}{a\Omega^2}$ $\Rightarrow$stable.

$$\theta = \theta_e + \epsilon \Rightarrow \ddot{\epsilon} + \Omega^2\sin^2\theta_e\epsilon = 0$$

Oscillatory Behavior.

Stability around $\theta_e = 0$

$\theta_e = 0 \Rightarrow$ consider small changes $\theta = \theta_e + \epsilon$, $\dot{\theta} = \dot{\epsilon}, \ddot{\theta} = \ddot{\epsilon}$

$$\ddot{\epsilon} - \epsilon\Omega^2 + \frac{g}{a}\epsilon = 0 \Rightarrow \ddot{\epsilon} + (\frac{g}{a} - \Omega^2)\epsilon = 0 \tag{2}$$

Ω^2: Controlled parameter. If Ω^2 is small, behavior is stable. If $\Omega^2 > \frac{g}{a}$, behavior is unstable.

Stable: $\Omega^2 < \frac{g}{a}$
Unstable: $\Omega^2 > \frac{g}{a}$

If we look for a solution to Equation 1 of the form $\epsilon = Ae^{\lambda t}$, we have:

$$\lambda^2 Ae^{\lambda t} + (\frac{g}{a} - \Omega^2)Ae^{\lambda t} = 0$$

$$\lambda = \pm\sqrt{(\Omega^2 - \frac{g}{a})}$$

If $\Omega^2 < \frac{g}{a}$, λ is imaginary $\Rightarrow$ oscillation.
If $\Omega^2 > \frac{g}{a}$, λ is real $\Rightarrow$ exponential growth.

Stability around $\theta_e = \pi$

$\theta_e = \pi$
$\theta = \theta_e + \epsilon = \pi + \epsilon$
From Equation (1)

$$\ddot{\epsilon} - \sin(\pi + \epsilon)\cos(\pi + \epsilon)\Omega^2 + \frac{g}{a}\sin(\pi + \epsilon) = 0$$

$$\sin(\pi + \epsilon) = \sin\pi\cos\epsilon + \cos\pi\sin\epsilon \approx -\epsilon$$

$$\cos(\pi + \epsilon) = \cos\pi\cos\epsilon - \sin\pi\sin\epsilon \approx -1$$

$$\ddot{\epsilon} - \epsilon\Omega^2 - \frac{g}{a}\epsilon = 0$$

$\ddot{\epsilon} - (\Omega^2 + \frac{g}{a})\epsilon = 0 \Rightarrow$ Unstable, because of the negative sign in front of the ϵ term.

Regime Diagram

Plot a regime diagram.

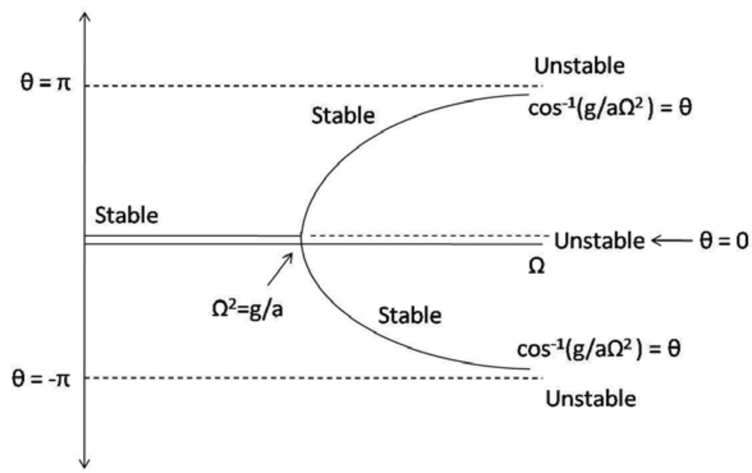

Figure 2: Regime diagram for modeled system. Depending on the the angle and angular velocity, the system may be in a stable or an unstable regime. Figure by MIT OCW.

The solutions are symmetrical around the x-axis: the mass can rise on either side. The solutions are symmetrical around the y-axis (Ω and $-\Omega$), because the hoop can spin clockwise or counterclockwise with the same behavior.

For $\theta_e = \pi, -\pi$, the equilibrium point is unstable for all Ω. For $\theta_e = 0$, the equilibrium point is stable until $\Omega^2 = \frac{g}{a}$. Beyond that Ω the point is unstable.

For $\theta_e = \arccos\left(\frac{g}{a\Omega^2}\right)$, the equilibrium point exists for $\frac{g}{a} = \Omega^2$ and then θ grows as Ω^2 increases. The equilibrium point is stable.

This system exhibits stability behavior known as a pitchfork bifurcation. At first, the mass oscillates about $\theta = 0$. As Ω is increased towards $\Omega^2 = g/a$ those oscillations continue until $\Omega^2 > g/a$. At that point, the mass rises to $\arccos g/a\Omega^2$ on one side of the hoop. θ continues to increase but it will not reach π or $-\pi$. Because perturbation can cause the mass to rise either at θ or $-\theta$, the behavior is called a bifurcation.

Alternative Method to Derive Linearized Perturbation Equations

This method makes approximations at the level of the Lagrangian. Then one differentiates to obtain linearized equations of motion. Do not use this method if there is any uncertainty about the equilibrium points. This method does not give the nonlinear equations of motion needed to find the equilibrium points.

Cart with Pendulum and Spring Example Revisited

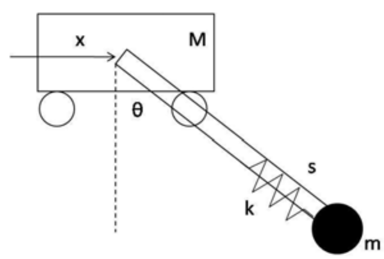

Figure 3: Cart with Pendulum and Spring. Figure by MIT OCW.

This example is explained in full in Lectures 16, 17, and 18.

$$L = \frac{1}{2}(M+m)\dot{x}^2 + \frac{1}{2}m(\dot{s}^2 + s^2\dot{\theta}^2 + 2\dot{x}(\dot{s}\sin\theta + s\dot{\theta}\cos\theta)) + mgs\cos\theta - \frac{1}{2}k(s-l)^2$$

Alternative Method:
1. Assume equilibrium solution is known.
2. Consider small changes to the variables.
3. Expand L up to its quadratic terms.
4. Use the approximate L in Lagrange's Equations to obtain linearized quations for stability analysis.

Assume Equilibrium Solution is Known

Equilibrium Solution: $\theta = 0$, $s_0 = l + \frac{mg}{k}$, $s = s_0 + \epsilon$, $x =$any value. Use 0 for this example.

This method helps if you already know all the equilibria.

Consider Small Changes to the Variables

$$
\begin{aligned}
L =& \frac{1}{2}(M+m)\dot{x}^2 + \frac{1}{2}m\left[\dot{\epsilon}^2 + (s_0+\epsilon)^2\dot{\theta}^2 + 2\dot{x}\left(\dot{\epsilon}\theta + (s_0+\epsilon)\dot{\theta}\left(1-\frac{\dot{\theta}^2}{2}\right)\right)\right] \\
&+ Mg(s_0+\epsilon)\left(1-\frac{\theta^2}{2}\right) - \frac{1}{2}k(s_0+\epsilon-l)^2
\end{aligned}
$$

Sine and Cosine Approximations

$\sin\theta \approx \theta - \frac{\theta^3}{3!}$

$\cos\theta \approx 1 - \frac{\theta^2}{2}$

Expand L up to its quadratic terms

$$
L = \frac{1}{2}(M+m)\dot{x}^2 + \frac{1}{2}(\dot{\epsilon}^2 + s_0^2\dot{\theta}^2 + 2\dot{x}s_0\dot{\theta}) + mgs_0 + mg\epsilon - mgs_0\frac{\theta^2}{2} - \frac{1}{2}k(s_0+\epsilon-l)^2
$$

$$
-\frac{1}{2}k(s_0+\epsilon-l)^2 = -\frac{1}{2}k(s_0-l)^2 - k(s_0-l)\epsilon - \frac{1}{2}k\epsilon^2
$$

Retained only quadratic nonlinearity in Lagrangian.
Keep only quadratic terms because when one differentiates, the quadratic terms will become linear terms.

Use Lagrange's Equations to Obtain Linearized Equations of Motion

x:

$$
\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \left(\frac{\partial L}{\partial x}\right) = 0
$$

$$
\left(\frac{\partial L}{\partial \dot{x}}\right) = (M+m)\dot{x} + ms_0\dot{\theta}
$$

$$
\left(\frac{\partial L}{\partial x}\right) = 0
$$

$$
\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = (M+m)\ddot{x} + ms_0\ddot{\theta} = 0 \Rightarrow \boxed{(M+m)\ddot{x} + ms_0\ddot{\theta} = 0}
$$

θ:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \left(\frac{\partial L}{\partial \theta}\right) = 0$$

$$\left(\frac{\partial L}{\partial \dot{\theta}}\right) = ms_0^2\dot{\theta} + ms_0\dot{s}$$

$$\left(\frac{\partial L}{\partial \theta}\right) = mgs_0\theta$$

$$ms_0^2\ddot{\theta} + ms_0\ddot{x} + mgs_0\theta = 0$$

$$\boxed{s_0\ddot{\theta} + \ddot{x} + g\theta = 0}$$

ϵ:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\epsilon}}\right) - \left(\frac{\partial L}{\partial \epsilon}\right)$$

$$\left(\frac{\partial L}{\partial \dot{\epsilon}}\right) = \dot{\epsilon}m$$

$$\left(\frac{\partial L}{\partial \epsilon}\right) = mg - k(s_0 + \epsilon - l)$$

$$m\ddot{\epsilon} - mg + ks_0 + k\epsilon - kl = 0$$

$$\boxed{m\ddot{\epsilon} + k\epsilon = 0}$$

Pitchfork Bifurcations

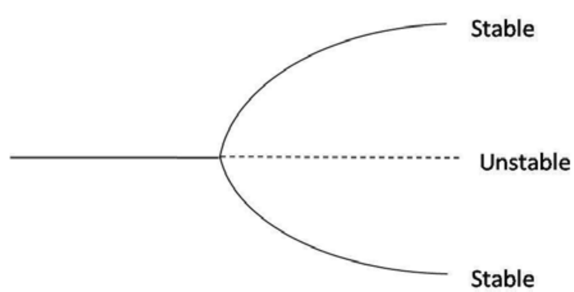

Figure 4: Bifurcation diagram. Figure by MIT OCW.

Only stable paths observable in the real world.

Changes in the control parameter determine what will happen.

Bifurcations: Turn up often in nonlinear dynamics.

Example: Rod Compression

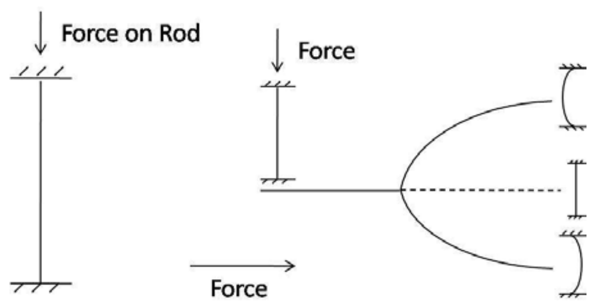

Figure 5: Rod compression. Rod buckles either to left or right. Configuration determined by initial perturbation after applying force.

Example: Liquid Crystal Displays

Liquid Crystal Display: Thin rodlike molecules. Back light gets through polarizer.

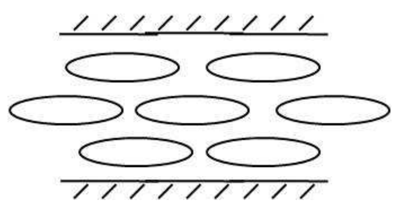

Figure 6: Schematic of liquid crystal display. Figure by MIT OCW.

Elastic forces that prevent reorientation.

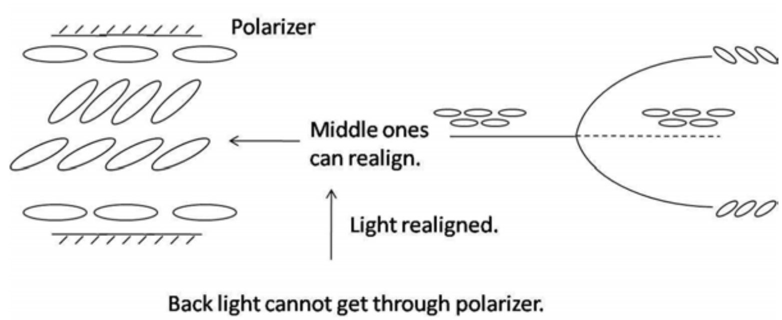

Figure 7: Schematic of liquid crystal display once light passes through. Figure by MIT OCW.