2.003J/1.053J Dynamics and Control I, Spring 2007
Professor Thomas Peacock
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Lecture 1

Newton's Laws, Cartesian and Polar Coordinates, Dynamics of a Single Particle

Big Picture

First Half of the Course → Momentum Principles (Force, Vectors) Newtonian Dynamics
Second Half of the Course → Lagrangian Dynamics (Energy, Scalar)

Both give equations of motion.

Restrictions:
a. Planar (2-D) Dynamics [(3-D) Dynamics 2.032 - Graduate follow on course]
b. Linear Dynamics [Nonlinear Dynamics I: Chaos 2.050J - Undergraduate follow on course]

Learning Objectives

You might comment, "I've seen this material before in 18.03, 8.01." You will use knowledge from previous courses including differential equations and mechanics.
After this course, you will be able to:

1. Apply knowledge to new problems

2. Define coordinate system

3. Obtain equations of motion

4. Solve equations

Isaac Newton's (1642 - 1727) Laws of Motion

Principia, Chapter 1, in Cambridge Trinity College. Originally in Latin.

I. 1st Law - A particle remains at rest or continues to move in a straight line with constant velocity if there is no resultant force acting on it.

II. 2nd Law - A particle acted upon by a resultant force moves in such a manner that the time rate of change of its linear momentum is equal to the force.

$$\sum \underline{F} = m\underline{a} \text{ for a single particle.}$$

where $\underline{F}$ is the force, m is the mass, and $\underline{a}$ is the acceleration.

III. 3rd Law - Forces that result from interactions of particles and such forces between two particles are equal in magnitude, opposite in direction, and collinear.

a. Read Williams Chapter 1 and 2 for background
   i. History of Calculus: Leibniz and Newton
   ii. Principia - Mostly geometric
   iii. Euler wrote $\sum \underline{F} = m\underline{a}$ and $\sum \underline{F} = \frac{d}{dt}(m\underline{v})$

Called laws because they have been tested. Cannot prove, but can disprove.

Inertial Frame of Reference

Newton II: Requires the concept of an "internal frame of reference" because $\frac{dp}{dt}$ depends on reference frame in which motion is observed (this is not so for mass of particle or the force applied to the particle.)

An internal reference frame must be non-accelerating. What is non-accelerating? (Newton I)

Absolute inertial frame is where Newton I holds and is an idealized situation. Surface of earth is good enough (for short periods of time: ~minutes). Earth is rotating. If particle followed long enough, will curve. (Fictitious force, coriollis effect).

Definitions of Velocity and Acceleration

The velocity of a particle B is the time rate of change of its position.

$$\underline{v} = \lim_{\Delta t \to 0} \frac{\Delta \underline{r}}{\Delta t} = \frac{d\underline{r}}{dt}$$

$\underline{r}$ is position, and t is time. The direction of $\underline{v}$ is in the direction of $\Delta\underline{r}$ as $\Delta t \to 0$.

The acceleration:

$$\underline{a} = \frac{d\underline{v}}{dt} = \frac{d^2\underline{r}}{dt^2}$$

Acceleration is the time rate of change of its velocity.

Two coordinate systems: Cartesian and Polar

Velocities and accelerations can be expressed using a variety of different coordinate systems. Here are two examples.

In Cartesian (rectangular) coordinates (x,y):

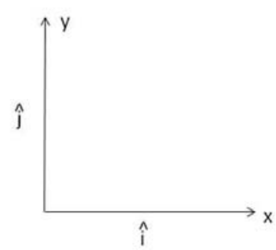

Figure 1: A Cartesian coordinate system. Figure by MIT OCW.

"ˆ" means unit vector; "·" means time derivative

$$\underline{r} = x\hat{\imath} + y\hat{\jmath}$$

$$\underline{v} = \dot{\underline{r}} = \dot{x}\hat{\imath} + \dot{y}\hat{\jmath}$$

$$\underline{a} = \dot{v} = \ddot{r} = \ddot{x}\hat{\imath} + \ddot{y}\hat{\jmath}$$

In polar coordinates (r,θ):

See Figure 2.

$$\underline{r} = r\hat{e_r}$$

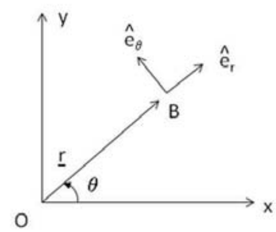

Figure 2: A polar coordinate system. r is the distance from origin. θ is the angle made with x-axis. $\hat{e_\theta}$ and $\hat{e_r}$ are perpendicular. Figure by MIT OCW.

$$\underline{v} = \frac{d\underline{r}}{dt} = \frac{dr}{dt}\hat{e_r} + r\frac{d\hat{e_r}}{dt} = \dot{r}\hat{e_r} + r\dot{\theta}\hat{e_\theta}$$

$$\underline{a} = \frac{d\underline{v}}{dt} = \ddot{r}\hat{e_r} + \dot{r}\dot{\theta}\hat{e_\theta} + \dot{r}\dot{\theta}\hat{e_\theta} + r\ddot{\theta}\hat{e_\theta} - r\dot{\theta^2}\hat{e_r}$$

$$\underline{a} = (\ddot{r} - r\dot{\theta^2})\hat{e_r} + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{e_\theta}$$

$$\frac{d}{dt}\hat{e_\theta} = -\dot{\theta}\hat{e_r}$$

Proof that $\frac{d\hat{e_r}}{dt} = \dot{\theta}\hat{e_\theta}$:

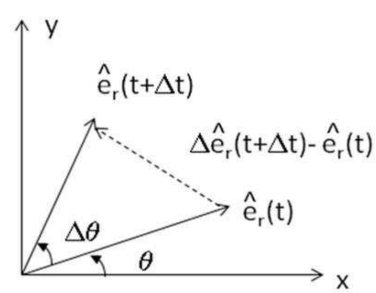

Figure 3: Differentiation of unit vectors. Changes in the direction of unit vector $\hat{e_r}$ can be related to changes in θ. Figure by MIT OCW.

$$\frac{d\hat{e_r}}{dt} = \lim_{\Delta t \to 0} \frac{\Delta\hat{e_r}}{\Delta t} = \lim_{\Delta t \to 0} \frac{|\hat{e_r}|\Delta\theta}{\Delta t}\hat{e_\theta} = \frac{d\theta}{dt}\hat{e_\theta} = \dot{\theta}\hat{e_\theta}$$

$$\hat{e_r} = \cos\theta\hat{\imath} + \sin\theta\hat{\jmath}$$

$$\frac{d}{dt}\hat{e_r} = -\sin\theta\dot{\theta}\hat{\imath} + \cos\theta\dot{\theta}\hat{\jmath} = \dot{\theta}(-\sin\theta\hat{\imath} + \cos\theta\hat{\jmath}) = \dot{\theta}\hat{e_\theta}$$

Dynamics of a Single Particle (Review)

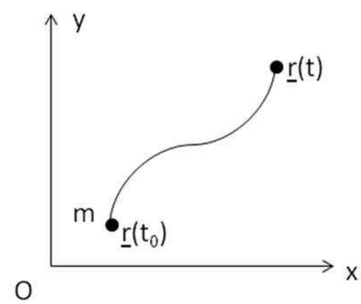

Figure 4: A point mass m moves from $\underline{r}(t_0)$ to $\underline{r}(t)$. Figure by MIT OCW.

Consequences of Newton's Second Law: Linear and Angular Momentum Conservation

Using an inertial frame of reference, here is the expression of Newton II:

$$\sum \underline{F} = m\underline{a}$$

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Linear Momentum Principle

$$\sum \underline{F} = \frac{d}{dt}(m\underline{v}) = \underline{\dot{p}} \qquad (\underline{\mathrm{p}} = \mathrm{m}\underline{\mathrm{v}} = \text{Linear Momentum})$$

If $\sum \underline{F} = 0$, $\underline{\mathrm{p}}$ is constant. (Conservation of Linear Momentum)

Angular Momentum Principle

Define Angular Momentum about B in an inertial frame of reference.

$\underline{h}_B = \underline{r} \times m\underline{v}$ is the moment of the linear momentum about B.

Investigate Properties (Take the Time Derivative)

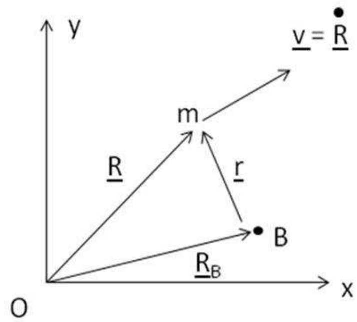

Figure 5: Vector relationships between moving point mass m and angular momentum around B. B is a general point (could be moving); choose B appropriately to help solve the problem. Figure by MIT OCW.

$$\begin{aligned}
\frac{d\underline{h}_B}{dt} &= \dot{\underline{r}} \times m\underline{v} + \underline{r} \times m\dot{\underline{v}} \\
&= (\dot{\underline{R}} - \dot{\underline{R}}_B) \times m\underline{v} + \underline{r} \times \sum \underline{F} \\
&= (\underline{v} \times m\underline{v}) - \underline{v}_B \times m\underline{v} + \underline{r} \times \sum \underline{F} \qquad (1) \\
&= -\underline{v}_B \times m\underline{v} + r \times \sum \underline{F} \qquad (2)
\end{aligned}$$

The first term in Equation (1) is zero, because the cross product of two parallel vectors is zero.

Define Resultant Torque around B. $\underline{\tau}_B = \underline{r} \times \sum \underline{F}$ is the moment of the total force about B. Combining this definition with Equation (2) yields

$$\underline{\tau}_B = \dot{\underline{h}}_B + \underline{v}_B \times m\underline{v}.$$

If $\tau_B = 0$ and $\underline{v}_B = 0$ or $v_B \parallel m\underline{v} \Rightarrow \dot{h}_B = 0$ (Conservation of Angular Momentum)