Massachusetts Institute of Technology
Department of Mechanical Engineering

2.003J/1.053J Dynamics & Control I

Fall 2007

Homework 8

Issued: Nov. 9. 2007 **Due**: Nov.30. 2007

Instructions (please read carefully) :

We have three nonlinear dynamics problems posted in this homework but please choose and solve only ONE that interests you. The hardcopy report should contain your written answer for each sub-question and **graphs** (no m-files). Supporting materials which are not included in your report (such as **m-files with appropriate comments)** should be also submitted on the MIT Server site. **Without the m-files, your answers will not be accepted.** There may be several places where you will have questions or get stuck. Please ask the TAs for any guidelines/ideas/clarifications.

Problem 8.1 : Nonlinear parametric pendulum

The equation of motion governing an undamped simple pendulum is given by:

$$\frac{d^2\theta}{dt^2} + \omega_0^2 \sin\theta = 0 \tag{1}$$

where $\omega_0 = \sqrt{\frac{g}{L}}$ is the natural frequency of the pendulum, g the acceleration due to gravity, L the length of the pendulum and θ the angle made with the vertical. Now, we can add a damping force to the equation by including a term that is proportional to the velocity of the pendulum. The equation of motion now becomes:

$$\frac{d^2\theta}{dt^2} + 2\gamma\frac{d\theta}{dt} + \omega_0^2 \sin\theta = 0 \tag{2}$$

where γ is the damping coefficient.

In this problem, we are interested in studying the nonlinear parametric pendulum, whose equation of motion is given by:

$$\frac{d^2\theta}{dt^2} + 2\gamma\frac{d\theta}{dt} + \omega_0^2\left[1 + h\cos\left(2\omega_0 t\right)\right]\sin\theta = 0 \tag{3}$$

The above is a simple model for a child playing on a swing. The forcing term models the periodic pumping of the child's legs at approximately twice the natural frequency of the swing. h is the amplitude of forcing. Let us see if we can answer the following question in this problem: starting near the equilibrium point $\theta = \dot{\theta} = 0$, can the child get the swing going by pumping his/her legs this way, or does he/she need a push ?

For this problem, choose $\omega_0 = 1$. Unless otherwise specified, use $\gamma = 0.1$. Also, remember that θ physically varies between 0 and 2π, and any value greater than 2π in the solution should be brought back to the physical range using the function `mod`.

i) For the unforced and undamped pendulum ($h = \gamma = 0$), find the time series $\theta(t)$ vs. t with the initial conditions:

   a) $\dot{\theta}(0) = 0, \theta(0) = 0.01$

   b) $\dot{\theta}(0) = 0, \theta(0) = 3$

   Show that the trajectory in a) is sinusoidal, whereas the trajectory in b) is not. This happens because, in part b), θ is so large that $\sin\theta \approx \theta$ is not a good approximation, and hence nonlinear effects kick in. Also, show that the time period of oscillations is dependent on the initial conditions for large θ (and $\dot{\theta}$), and is independent of the initial conditions for small θ (and $\dot{\theta}$).

ii) The rest state $\theta = \dot{\theta} = 0$ of the pendulum is known to be unstable when $\gamma^2 < \left(h\omega_0\right)^2/16$. Verify the prediction of the critical value of h necessary for sustained oscillations of the pendulum. (Get the time series for two values of h, one that is just below and the other that is just above the critical value of h given by the instability condition.)

iii) Plot the time series $\theta(t)$ (for any initial condition) for h =1.50, 1.80, 2.00, 2.05, 2.062. Calculate the time periods and comment on the pattern you observe. (Remember to let the initial transients die away before you estimate the time period.)

iv) The motion of the pendulum for h =2.2 is on a "strange attractor." An attractor (a set of points to which solutions get attracted) is informally described as strange if the dynamics on the attractor are chaotic. When a system gets into the chaotic regime, the behavior of the solution will seem random even though we have a deterministic system that governs the motion of the pendulum. For h =2.2, plot the trajectory on a $\theta-\dot{\theta}$ plane, and observe the chaotic (random) nature of the "strange attractor"! (Use any non-zero initial condition; you can also plot the time series of θ or $\dot{\theta}$ and observe that the behavior is random and there is no clear time period in the signal.) Again, remember to let the initial transients die away before you make your plots. Unpredictability is an essential feature of chaos. If a system is in the chaotic regime, it is impossible to predict the exact position and velocity of the pendulum at a future time even if we are given the current position and velocity with maximum accuracy. This property of unpredictability arises from what we call "sensitive dependence on initial conditions." We may start the system at (0.1,0.1) and observe a specific trajectory; but, starting at (0.1,0.10000001) may result in a completely different trajectory if the behavior is chaotic. Show that there is sensitive dependence on initial conditions for h =2.2. (For showing sensitive dependence on initial conditions, consider two initial conditions that are different only by a very small quantity (something like $\delta\theta$ = 0.001 radians should do the job), and show that the solutions are very different after some time when plotted on top of each other.)

v) Finally, make a quick comparison of the time series of θ for h =1.8, which is well below the critical value for chaos, and h =2.2 when the system is chaotic. Use the same initial conditions for the two cases.

Problem 8.2 : The growth/decay of population of animal species

A popular model for the growth/decay of population of animal species is given by the logistic map. It takes into account the two basic factors that contribute to the birth or death of any animal species:

a) *Reproduction* means the population will increase at a rate proportional to the current population. This can be stated as "the population at a future time is proportional to the population at the current time," i.e.

$$x_{n+1} \propto x_n \tag{4}$$

where x_{n+1} is the population at time $n+1$ given that the population at time n is x_n. The proportionality constant signifies the growth rate because of reproduction.

b) *Starvation* means the population will decrease at a rate proportional to the value obtained by taking the theoretical "carrying capacity" of the environment less the current population. In a given environment, the presence of more number of animals will result in more rapid consumption of the available resources, and this will result in more likelihood of starvation. This can be written as

$$x_{n+1} \propto (1 - x_n) \tag{5}$$

Since x_n and x_{n+1} are normalized populations (actual population divided by the maximum population possible), they cannot be greater than 1. $(1 - x_n)$ is a measure of how close the system is to the maximum carrying capacity of the environment. The proportionality constant signifies the inverse of the death rate because of starvation.

Mathematically, the complete model that includes both the birth and the death processes can be written as:

$$x_{n+1} = r x_n (1 - x_n) \tag{6}$$

where x_n is a number between zero and one, and represents the population at year n; hence x_0 represents the initial population (at year 0); r is a positive number, and represents a combined rate for reproduction and starvation.

i) For values of r less than 1, show that, irrespective of the initial population, the population will eventually die.

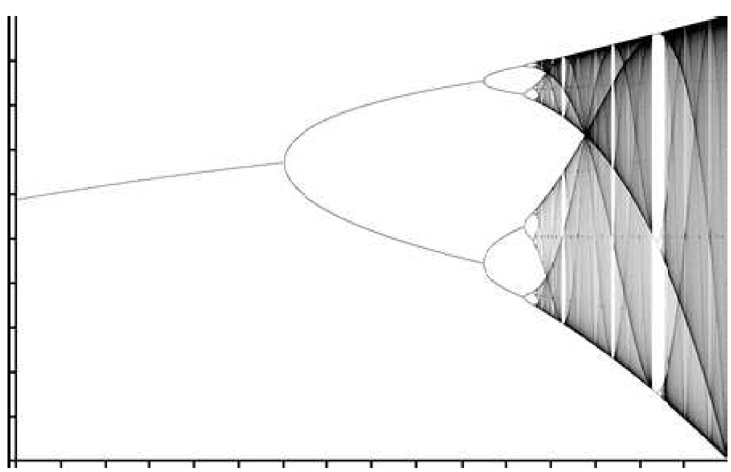
ii) With r between 1 and 2, show that the population quickly stabilizes to the value $\frac{r-1}{r}$, independent of the initial population.

iii) What happens for values of r between 2 and 3 ?

iv) Play around with the value of r, and see if you can find a critical value above which the solution no more converges to $\frac{r-1}{r}$. Plotting x_n vs. n should reveal that the solution never settles to a single value even for large n. Instead of converging to a single value, the solution converges to a 2-period oscillation or a 4-period oscillation, or some 2^k-period oscillation as you increase r. At the first critical value of r, the solution will switch from settling to a single value to settling to an oscillation between two values; similarly, at the second critical value of r, the solution will switch from settling to an oscillation between two values to settling to an oscillation between four values; and so on. This is known as period-doubling. As we can guess, for large enough r, the solution will converge to an oscillation between 2^k values, where k is large, and effectively, there is no finite-period oscillation to which the system settles, and this is known as the "period-doubling route to chaos".

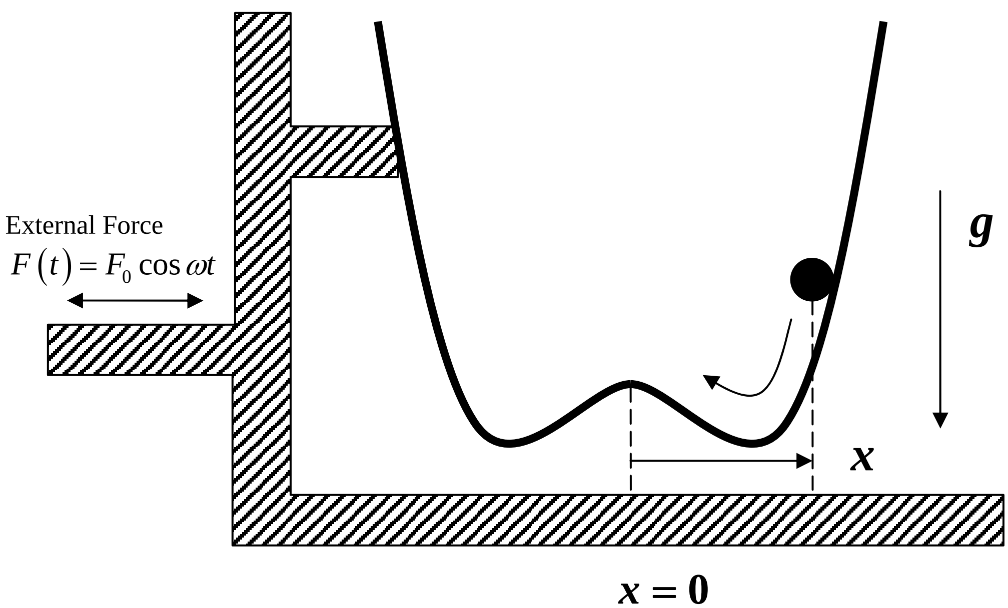
Show sample plots of x_n Vs n for various values of r to confirm that period-doubling happens. Find the first four values of r where period doubling occurs.

v) Create a rough map of the asymptotic values of x_n for varying r. (What we are looking for is the following: On the x-axis, you should plot r. On the y-axis should be the asymptotic values to which the population converges for the corresponding value of r. The asymptotic values to which the population converges will be either 1 or 2 or 4 or 8 or some 2^k in number depending on the value of r. You can do something like getting the solution for a large number of iterations (say 50000) and plot the last 10000 values. If the solution converges to a single value, we can expect the last 10000 values to be the same; if the solution converges to a 2-period oscillation, we can expect the last 10000 values to be switching between the two values of the 2-period oscillation, and so plotting the last 10000 values is equivalent to plotting just the two values in the oscillation cycle; See the below figure which your rough map should look like!).



vi) At $r = 3.57$ (approximately) is the onset of chaos, at the end of the period-doubling cascade. We can no longer see any oscillations. Take $r = 3.7$, and show that there is now sensitive dependence on initial conditions. (Slight variations in the initial population yield dramatically different results over time, a prime characteristic of chaos. For more details on unpredictability and sensitive dependence on initial conditions, see part iv) of the previous question.)

Problem 8.3 : Double-Well Potential System



The above figure shows a double well potential system. The double well is attached to a rigid framework and the particle moves along the double well. As seen in the figure, the two wells are separated by a hump at $x = 0$ has two stable states where their potentials are minimal. So, it is called bistable system. The potential energy is proportional to the height of the particle. The double well potential can be expressed as $V(x) = \frac{1}{4}x^4 - \frac{1}{2}x^2$. Suppose the double well is shaken periodically by the external force from side to side. What happens to the motion of the damped particle? If the shaking is very weak, the particle will go into one of the wells and stay near the bottom of the well, jiggling slightly. With strong shaking, the particle trajectory will become larger. At this moment, we can guess that two types of particle motion exist: a small-amplitude, low-energy oscillation about the bottom of a well, and a large-amplitude, high-energy oscillation where the particle goes back and forth over the hump without settling down in one of the wells. Your task is to observe the particle motion under Stokes type damping with coefficient δ as a function of initial conditions and the magnitude of external force. The motion equation of the particle along the double well is given as follows:

$$\ddot{x}+\delta\dot{x}-x+x^3=F_0\cos\omega t \tag{8}$$

In this problem, the damping coefficient and the frequency of the external force are given: $\delta=0.25$ & $\omega=1$.

i) Suppose that small external force $F=0.18$ is applied. Show that the motion of the particle depends on the initial conditions. (You can choose different initial conditions, and calculate their trajectory, and compare together. 100-200 sec is enough to simulate the motion.) Confirm that all the trajectories of the particle are oscillatory and confined to single well at the steady state. Is there any change of the period for different initial conditions?

ii) Suppose that large external force $F=0.4$ is applied. Compare the particle trajectories for different initial conditions. Are they relevant? Irrelevant? Or chaotic? Describe what you observe (for the definition of chaos, see problem 1).

iii) Now, we introduce the phase plane analysis. Phase plane is sometimes very useful to analyze the nonlinear dynamic motions. Phase plane is defined as the plot of $x(t)$ (x axis) vs. $\dot{x}(t)$(y axis), and you can observe dynamic behavior, independent of time. Generate phase plane plot for different initial conditions you used in ii) (separate plots), and compare them together. (Use simulation time of 1000-2000 sec to observe the behavior well.)

iv) From iii), you may recognize phase plane analysis is hard to interpret. At this time, another nonlinear dynamic analysis tool is suggested, Poincaré section. It is obtained by plotting $x(t)$ vs $\dot{x}(t)$ for the same phase. (Since the external force is periodic, you obtain time values when they have same phase. For example, the phase of the external force at t=0, 2π/ω, 4π/ω, …… are same.). From a physical view point, we take "snapshot" of the system at the same phase in each driver cycle. Plot Poincaré section for the initial conditions used in ii). (Plot the points, not line between two points in the figures. 10000-20000 sec simulation time is enough to generate Poincaré section.) Is it better than phase plane analysis in terms of visualization? Describe what you observe for different initial conditions.

v) In general, chaos still happens even when time goes to the infinity, but this system has different type of chaos: with two initial conditions which are close together, their trajectories aren't similar even in the transient response. However, either trajectory eventually goes to a steady state, depending on particle's transient behavior. It is called transient chaos, and chaos happens only in the transient response. To find suitable initial conditions, trial and error is used. Plot the trajectories of particle separately for following initial conditions.

a. $x(0) = 0.177 \quad \& \quad \dot{x}(0) = 0.1$

b. $x(0) = 0.176 \quad \& \quad \dot{x}(0) = 0.1$

Use $F = 0.25$ as the magnitude of external force and 500 sec as the simulation time. Describe what you observe by comparing both trajectories.

vi) Now, you understand it can be very hard to predict the final state of this system depending on the initial condition. Therefore, the sensitivity of initial condition is conveyed more vividly by the following graphical method.

a. Initial conditions for position and velocity should be $-2.5 \le x(0), \dot{x}(0) \le 2.5$

b. Make a 101 x 101 color grids with each grid point representing the particle's final state with the given initial condition.

c. Calculate trajectory up to 500 second for each initial condition.

d. A red dot is placed in a corresponding grid point if the trajectory ends up in the right well.

e. Blue dot is placed in a corresponding grid point if the trajectory ends up in the left well.

Plot the color-coded grids for different initial conditions. Is it beautiful? (Hint: you plot the map with `'imagesc'` function. This function requires matrix (grid) to be imaged in the figure. You define empty matrix for color coding. If color is red in corresponding grid, matrix element has 1. Otherwise, it has -1.)