MASSACHUSETTS INSTITUTE OF TECHNOLOGY
DEPARTMENT OF MECHANICAL ENGINEERING
CAMBRIDGE, MA 02139

2.002 MECHANICS AND MATERIALS II

Spring, 2004

Creep and Creep Fracture: Part II
Stress and Deformation Analysis in Creeping Structures

Steady-State Bending of Viscoplastic Beams

1. Kinematics

$v(x,t)$	trans. displ. of the neutral axis
$\kappa(x,t) \doteq \frac{\partial^2 v(x,t)}{\partial x^2}$	curvature of the neutral axis
$\epsilon(x,y,t) = -\kappa(x,t)y$	longitudinal strain
$\dot{\epsilon}(x,y,t) - \dot{\kappa}(x,t)y$	longitudinal strain rate
$\dot{\kappa}(x,y,t) \doteq \frac{\partial^2 \dot{v}(x,t)}{\partial x^2}$	curvature rate

$$
\begin{aligned}
|\dot{\epsilon}|\,\mathrm{sgn}(\dot{\epsilon}) &= -|\dot{\kappa}|\,\mathrm{sgn}(\dot{\kappa})|y|\,\mathrm{sgn}(y) \\
&= |\dot{\kappa}||y|\left[-\mathrm{sgn}(\dot{\kappa})\mathrm{sgn}(y)\right] \\
\text{Therefore, } |\dot{\epsilon}| &= |\dot{\kappa}||y| \text{ and} \\
\mathrm{sgn}(\dot{\epsilon}) &= -\mathrm{sgn}(\dot{\kappa})\mathrm{sgn}(y)
\end{aligned}
$$

2. Constitutive Relation

$$
\begin{aligned}
\dot{\epsilon} = \dot{\epsilon}^c &= \dot{\epsilon}_0 \left\{ \frac{|\sigma|}{s} \right\}^n \mathrm{sgn}(\sigma) \\
\sigma &= s \left\{ \frac{|\dot{\epsilon}|}{\dot{\epsilon}_0} \right\}^{1/n} \mathrm{sgn}(\dot{\epsilon}) \\
\Rightarrow \sigma(x, y, t) &= s \left\{ \frac{|\dot{\kappa}(x, t)||y|}{\dot{\epsilon}_0} \right\}^{1/n} \mathrm{sgn}(\dot{\epsilon}) \\
\Rightarrow \sigma(x, y, t) &= \{-\mathrm{sgn}(\dot{\kappa})\mathrm{sgn}(y)s\} \left(\frac{|\dot{\kappa}(x, t)|}{\dot{\epsilon}_0} \right)^{1/n} |y|^{1/n}
\end{aligned}
$$

$\dot{\epsilon}_0$	reference strain rate
$s > 0$	reference stress

3. Moment-Curvature Rate Relation

$$\begin{aligned} M(x,t) &= -\int_A y\sigma(x,y,t)dA \\ &= \{\mathsf{sgn}(\dot{\kappa})s\}\left(\frac{|\dot{\kappa}(x,t)|}{\dot{\epsilon}_0}\right)^{1/n}\int_A \mathsf{sgn}(y)|y|^{1+1/n}dA \\ \text{with}\quad I_n &\equiv \int_A \mathsf{sgn}(y)\,|y|^{1+1/n}\,dA \\ M(x,t) &= \{\mathsf{sgn}(\dot{\kappa})s\}\left(\frac{|\dot{\kappa}(x,t)|}{\dot{\epsilon}_0}\right)^{1/n} I_n \end{aligned}$$

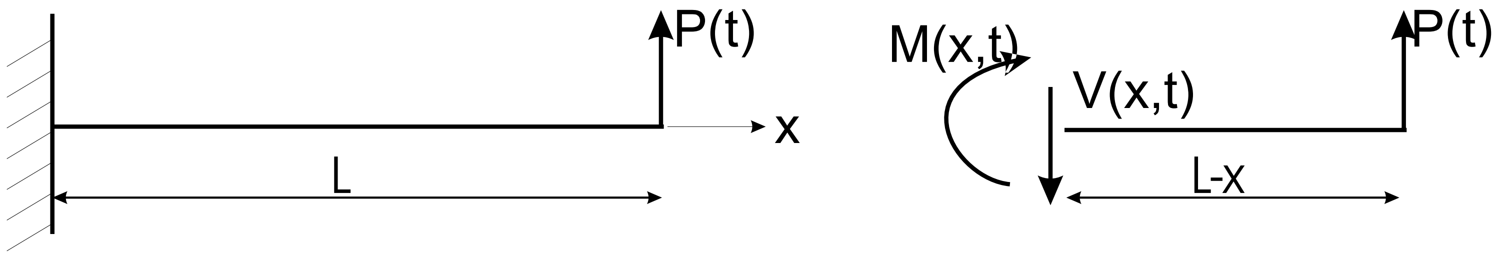
4. Equation for Stress

$$
\begin{aligned}
\sigma(x,y,t) &= -\{\mathsf{sgn}(\dot{\kappa})s\}\left(\frac{|\dot{\kappa}(x,t)|}{\dot{\epsilon}_0}\right)^{1/n}|y|^{1/n}(\mathsf{sgn}(y)) \\
\Rightarrow \sigma(x,y,t) &= -\frac{M(x,t)}{I_n}|y|^{1/n}\mathsf{sgn}(y)
\end{aligned}
$$

5. Differential Equation for Lateral Displacement

$$
\begin{aligned}
|M(x,t)| &= \left(\frac{|\dot{\kappa}(x,t)|}{\dot{\epsilon}_0}\right)^{1/n} sI_n \\
\Rightarrow \dot{\kappa}(x,t) &= \dot{\epsilon}_0\left\{\frac{|M(x,t)|}{sI_n}\right\}^n \mathsf{sgn}(M(x,t)) \\
\frac{\partial^2 \dot{v}(x,t)}{\partial x^2} &= \dot{\epsilon}_0\left\{\frac{|M(x,t)|}{sI_n}\right\}^n \mathsf{sgn}(M(x,t))
\end{aligned}
$$

Example Problem: Cantilever Beam



$$
\begin{aligned}
M(x,t) - P(t)(L-x) &= 0 \\
M(x,t) &= P(t)(L-x) \;\; 0 \le x \le L \\
\dot{\kappa}(x) = \frac{\partial^2 \dot{v}}{\partial x^2} &= \dot{\epsilon}_0 \left\{ \frac{|M|}{sI_n} \right\}^n \mathrm{sgn}(M) \\
&= \dot{\epsilon}_0 \left\{ \frac{|P(t)(L-x)|}{sI_n} \right\}^n \\
&= \dot{\epsilon}_0 \left\{ \frac{|P(t)|}{sI_n} \right\}^n (L-x)^n
\end{aligned}
$$

Example Problem: Cantilever Beam (cont.)

Boundary conditions: (1) $\dot{v} = 0$ at $x = 0$ and (2) $\frac{\partial \dot{v}}{\partial x} = 0$ at $x = 0$ (Assume $P(t) > 0$)

$$
\begin{aligned}
\frac{\partial \dot{v}}{\partial x} &= \dot{\epsilon}_0 \left\{ \frac{|P(t)|}{sI_n} \right\}^n \left\{ -\frac{1}{n+1} \right\} (L-x)^{n+1} + C_1 \\
C_1 &= \dot{\epsilon}_0 \left\{ \frac{|P(t)|}{sI_n} \right\}^n \frac{L^{n+1}}{n+1} \qquad \text{(Using BC (2))} \\
\frac{\partial \dot{v}}{\partial x} &= \dot{\epsilon}_0 \left\{ \frac{|P(t)|}{sI_n} \right\}^n \frac{1}{n+1} \left[-(L-x)^{n+1} + L^{n+1} \right] \\
\dot{v} &= \dot{\epsilon}_0 \left\{ \frac{|P(t)|}{sI_n} \right\}^n \frac{1}{n+1} \left[\frac{(L-x)^{n+2}}{n+2} + L^{n+1} x \right] + C_2 \\
C_2 &= -\dot{\epsilon}_0 \left\{ \frac{|P(t)|}{sI_n} \right\}^n \frac{1}{n+1} \left[\frac{L^{n+2}}{n+2} \right] \qquad \text{(Using BC (1))}
\end{aligned}
$$

Example Problem: Cantilever Beam (cont.)

$$
\begin{aligned}
\dot{v} &= \dot{\epsilon}_0 \left\{ \frac{|P(t)|}{sI_n} \right\}^n \frac{1}{n+1} \left[\frac{(L-x)^{n+2}}{n+2} + L^{n+1}x - \frac{L^{n+2}}{n+2} \right] \\
\dot{\delta} &= |\dot{v}(x=L)| = \dot{\epsilon}_0 \left\{ \frac{|P(t)|}{sI_n} \right\}^n \frac{L^{n+2}}{n+2} \quad \text{(Tip deflection rate)} \\
|P(t)| &= \left[\frac{(\dot{\delta}/\dot{\epsilon}_0)(n+2)}{L^{n+2}} \right]^{1/n} sI_n
\end{aligned}
$$

Three-Dimensional Generalization of Constitutive Equations for Elastic-Viscoplastic Materials

1. Strain Rate Decomposition:

$$\dot{\epsilon}_{ij} = \dot{\epsilon}^{e}_{ij} + \dot{\epsilon}^{c}_{ij}$$

$\dot{\epsilon}_{ij}$	total strain rate
$\dot{\epsilon}^{e}_{ij}$	elastic strain rate
$\dot{\epsilon}^{c}_{ij}$	creep or viscoplastic strain rate

2. Constitutive Equations for $\dot{\epsilon}^{e}_{ij}$:

$$\dot{\epsilon}^{e}_{ij} = \frac{1}{E}\left[(1+\nu)\dot{\sigma}_{ij} - \nu\left(\sum_{k}\dot{\sigma}_{kk}\right)\delta_{ij}\right]$$

E	Young's modulus
ν	Poisson's ratio

2. Constitutive Equations for $\dot{\epsilon}^c_{ij}$:

$$\dot{\epsilon}^c_{ij} = \dot{\overline{\epsilon}}^c (3/2)\{\sigma'_{ij}/\overline{\sigma}\} \quad \text{creep strain rate components}$$

$$\dot{\overline{\epsilon}}^c = \dot{\epsilon}_0 \{\overline{\sigma}/s\}^n \quad \text{equivalent tensile creep rate}$$

$$\sigma'_{ij} = \sigma_{ij} - (1/3)\left(\sum_k \sigma_{kk}\right)\delta_{ij} \quad \text{stress deviator components}$$

$$\overline{\sigma} = \sqrt{(3/2)\sum_{i,j}\sigma'_{ij}\sigma'_{ij}} \quad \text{Mises equivalent tensile stress}$$

$$= \left|(1/2)\left\{(\sigma_{11}-\sigma_{22})^2 + (\sigma_{22}-\sigma_{33})^2 + (\sigma_{33}-\sigma_{11})^2\right\} + 3\left\{\sigma_{12}^2 + \sigma_{23}^2 + \sigma_{31}^2\right\}\right|^{1/2}$$

n	creep exponent
$\dot{\epsilon}_0$	reference strain rate
s	reference stress

- Note that in uniaxial tension when $\sigma_{11} = \sigma$, with all other $\sigma_{ij} = 0$, we have $\sigma'_{11} = (2/3)\sigma$, $\sigma'_{22} = \sigma'_{33} = -(1/3)\sigma$, and $\overline{\sigma} = |\sigma|$. Therefore, the constitutive equation for $\dot{\epsilon}^c_{ij}$ yields

$$\dot{\epsilon}^c_{11} = \dot{\epsilon}_0 \{\overline{\sigma}/s\}^n \,\mathrm{sgn}(\sigma)$$

$$\dot{\epsilon}^c_{22} = \dot{\epsilon}^c_{33} = -(1/2)\dot{\epsilon}^c_{11}$$

$$\dot{\epsilon}^c_{ij} = 0 \text{ otherwise,}$$

as it should.

- For the case of **rigid-viscoplastic** materials, the elastic strains and strain rates are neglected:

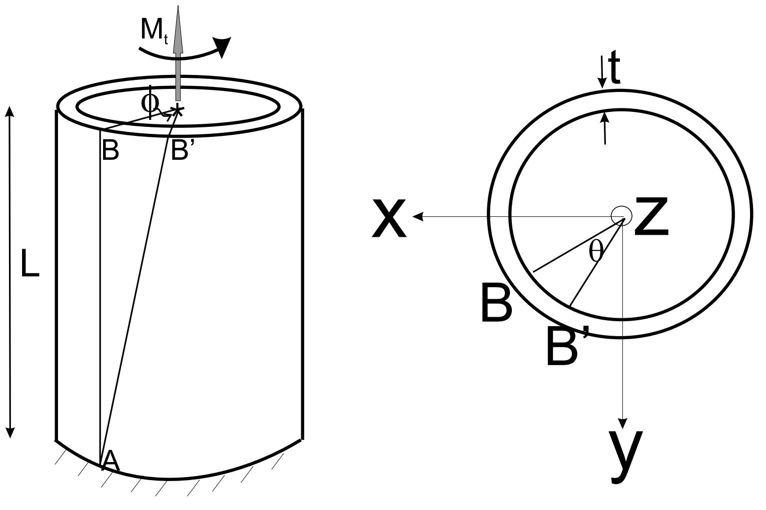
$$\epsilon_{ij} \doteq \epsilon_{ij}^c;$$

$$\dot{\epsilon}_{ij} = \dot{\epsilon}_{ij}^c = \dot{\overline{\epsilon}}^c (3/2)\{\sigma'_{ij}/\overline{\sigma}\}$$

$$\dot{\overline{\epsilon}}^c = \dot{\epsilon}_0 \{\overline{\sigma}/s\}^n$$

Example Problem: Torsion of Thin-walled Tube

- Consider a thin-walled tube of radius r, wall thickness t and length L. One end of the tube is fixed, while on the other a constant twisting moment M_t is applied. The tube is at high homologous temperatures (creep conditions prevail). Calculate the twisting rate $\dot{\phi}$ for the tube.



Example Problem: [Thin-Walled] Torsion

- The angle of twist ϕ is a function of time, i.e., $\phi = \phi(t)$. The angle of twist per unit length is denoted by $\alpha = \phi/L = \alpha(t)$

- Displacement field:

$$u_r = 0$$

$$u_\theta = \alpha z r$$

$$u_z = 0$$

- Strain field:

$$\epsilon_{\theta z} = \frac{1}{2}\left[\frac{\partial u_\theta}{\partial z} + \frac{1}{r}\frac{\partial u_z}{\partial \theta}\right] = \frac{1}{2}\alpha r,$$

with all other $\epsilon_{ij} = 0$

- Therefore the strain rate is

$$\begin{aligned} \dot{\epsilon}_{\theta z} &= \frac{1}{2}\dot{\alpha} r \\ \dot{\epsilon}_{\theta z} &= \frac{1}{2}\frac{\dot{\phi}}{L} r \end{aligned} \qquad (1)$$

- <u>Constitutive equation:</u> Since the applied moment $M(t)$ is constant, the elastic strain rate $\dot{\epsilon}^e_{ij} = 0$. (Strictly, we need to verify that $\dot{\sigma}_{ij} = 0$; however, the thin-walled tube in torsion has one constant non-zero stress component, $\sigma_{\theta z}$, that is directly proportional to twisting moment, M_t (see 'Equilibrium', below)). Therefore,

$$\dot{\epsilon}_{ij} = \dot{\overline{\epsilon}}^c \frac{3}{2}\left\{\frac{\sigma'_{ij}}{\overline{\sigma}}\right\} \Rightarrow \dot{\epsilon}_{\theta z} = \dot{\overline{\epsilon}}^c \frac{3}{2}\left\{\frac{\sigma'_{\theta z}}{\overline{\sigma}}\right\}$$

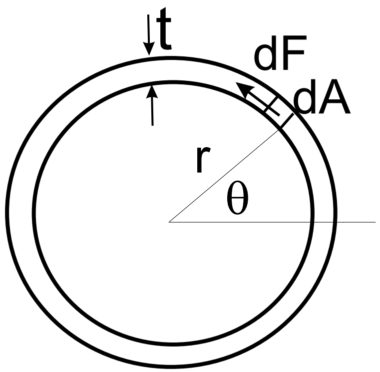
$$\dot{\overline{\epsilon}}^c = \dot{\epsilon}_0 \left\{\frac{\overline{\sigma}}{s}\right\}^n$$

$$\sigma'_{ij} = \sigma_{ij} - \frac{1}{3}\left(\sum_k \sigma_{kk}\right)\delta_{ij}$$

- The only non-zero stress component is $\sigma'_{\theta z} = \sigma_{\theta z}$. Also, $\overline{\sigma} = \sqrt{3}\,|\sigma_{\theta z}|$ from the definition of the equivalent tensile stress. Therefore,

$$\dot{\epsilon}_{\theta z} = \frac{\sqrt{3}}{2}\dot{\epsilon}_0 \left\{ \frac{\sqrt{3}\,|\sigma_{\theta z}|}{s} \right\}^n \mathrm{sgn}(\sigma_{\theta z}) \qquad (2)$$

- Equilibrium: The applied torque should balance the internal torque of the only non-zero stress component, $\sigma_{\theta z}$:



$$dM_t = rdF = r\sigma_{\theta z}\,dA = r\sigma_{\theta z}(rd\theta t) = \sigma_{\theta z}tr^2\,d\theta$$

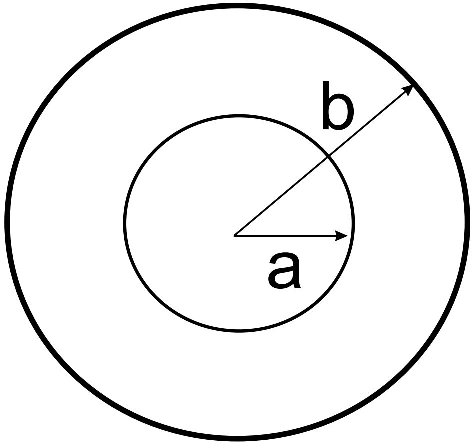
$$M_t = \int_0^{2\pi} \sigma_{\theta z}tr^2\,d\theta = 2\pi tr^2\sigma_{\theta z}$$

$$\Rightarrow \sigma_{\theta z} = \frac{M_t}{2\pi tr^2} \qquad (3)$$

- Finally,

$$
\begin{aligned}
\dot{\epsilon}_{\theta z} &= \frac{\sqrt{3}}{2}\dot{\epsilon}_0 \left\{ \frac{\sqrt{3}\,|\sigma_{\theta z}|}{s} \right\}^n \mathrm{sgn}(\sigma_{\theta z}) \\
\Rightarrow \frac{1}{2}\frac{\dot{\phi}}{L} r &= \frac{\sqrt{3}}{2}\dot{\epsilon}_0 \left\{ \frac{\sqrt{3}}{s} \frac{|M_t|}{2\pi t r^2} \right\}^n \mathrm{sgn}(M_t) \\
\Rightarrow \dot{\phi} &= \sqrt{3}\dot{\epsilon}_0 \left[\frac{\sqrt{3}|M_t|}{2\pi t r^2\, s} \right]^n \left(\frac{L}{r} \right) \mathrm{sgn}(M_t) \\
\Rightarrow \dot{\phi} &= \sqrt{3}\dot{\epsilon}_0 \left[\frac{\sqrt{3} M_t}{2\pi t r^2\, s} \right]^n \left(\frac{L}{r} \right) \quad \text{for } M_t > 0
\end{aligned}
$$

Example Problem: Torsion of Thick-walled Tube



Tube Cross-section

- Recall

$$\frac{1}{2}\,\dot{\alpha}r = \dot{\epsilon}_{\theta z}(r) = \frac{\sqrt{3}}{2}\,\dot{\epsilon}_0\,\left\{\frac{\sqrt{3}\,|\sigma_{\theta z}(r)|}{s}\right\}^n \mathrm{sgn}(\sigma_{\theta z}(r))$$

(for $a \leq r \leq b$)

For simplicity, let $\sigma_{\theta z}(r) > 0$. Therefore,

$$\dot{\epsilon}_{\theta z} = \frac{\sqrt{3}}{2}\,\dot{\epsilon}_0\,\left\{\frac{\sqrt{3}\,|\sigma_{\theta z}|}{s}\right\}^n \Rightarrow \sigma_{\theta z}(r) = \frac{s}{\sqrt{3}}\left\{\frac{2}{\sqrt{3}\dot{\epsilon}_0}\dot{\epsilon}_{\theta z}(r)\right\}^{1/n};$$

$$\dot{\epsilon}_{z\theta}(r) = \dot{\alpha}r/2 \Rightarrow \sigma_{z\theta}(r) \equiv Ar^{1/n},$$

where

$$A \equiv \frac{s}{\sqrt{3}}\left\{\frac{\dot{\phi}}{\sqrt{3}\dot{\epsilon}_0 L}\right\}^{1/n}$$

With $dM_t = \sigma_{\theta z}(r)r^2\,dr\,d\theta = Ar^{2+1/n}\,dr\,d\theta$,

$$M_t = A\int_a^b\int_0^{2\pi} r^{2+1/n}d\theta dr$$

Let

$$J_n = \int_a^b\int_0^{2\pi} r^{2+1/n}d\theta dr = \frac{2\pi}{3+1/n}\left[b^{3+1/n} - a^{3+1/n}\right]$$

Since $A = M_t/J_n$,

$$\sigma_{\theta z}(r) = \frac{M_t}{J_n} r^{1/n}$$

and

$$\frac{s}{\sqrt{3}} \left\{ \frac{\dot{\phi}}{\sqrt{3}\dot{\epsilon}_0 L} \right\}^{1/n} = \frac{M_t}{J_n}$$

$$\Rightarrow \dot{\phi} = \sqrt{3}\dot{\epsilon}_0 \left[\frac{\sqrt{3}\, M_t}{s\, J_n} \right]^n L$$