The Polynomial Method, Fall 2012, Problem Set 4

1. Go back to the summary you made for the proof of the 3D analogue of Szemerédi-Trotter. Just using the summary, try to reconstruct the proof. (If you want to try doing it without looking at your summary, that's good too.) You don't have to turn anything in. Don't be worried if you have some trouble. If you do have some trouble, write me a couple of sentences about how it went. In particular, can you find a particular mathematical question or strategic question that it would help to understand?

2. Let γ be a strictly convex closed curve in $B^2(R) \subset \mathbb{R}^2$. Prove that the number of lattice points lying on the curve γ is $\lesssim R^{2/3}$.

Hint: consider translates of γ and use the crossing number theorem.

3. A "grid" of algebraic curves. Suppose that $\alpha_i : \mathbb{R} \to \mathbb{R}^3$ are polynomials of degree $\leq d$ with disjoint images. Suppose that $\beta_j : \mathbb{R} \to \mathbb{R}^3$ are polynomials of degree $\leq d$ with disjoint images. Suppose that for each i, j, the image of α_i intersects the image of β_j. Prove that all the images are contained in a polynomial surface $Z(P)$ with $deg(P) \leq 100d^2$.

4. (optional) Imagine you are talking about math in a coffee shop with a friend. You tell your friend about Thue's theorem. Perhaps you give an example: the equation $2x^3 - y^3 = 15$ has only finitely many integer solutions. Your friend works in another branch of math, and they've never heard of Thue's work. They say, "how can somebody get started on that kind of problem? Suppose there were infinitely many integer solutions. What could we do with them?" What would you say?

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