

Lecture 9: Volatility Modeling

MIT 18.S096

Dr. Kempthorne

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Outline

- 1 Volatility Modeling
 - Defining Volatility
 - Historical Volatility: Measurement and Prediction
 - Geometric Brownian Motion
 - Poisson Jump Diffusions
 - ARCH Models
 - GARCH Models

Defining Volatility

Basic Definition

- Annualized standard deviation of the change in price or value of a financial security.

Estimation/Prediction Approaches

- Historical/sample volatility measures.
- Geometric Brownian Motion Model
- Poisson Jump Diffusion Model
- ARCH/GARCH Models
- Stochastic Volatility (SV) Models
- Implied volatility from options/derivatives

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Historical Volatility

Computing volatility from historical series of actual prices

- Prices of an asset at $(T + 1)$ time points

$$\{P_t, t = 0, 1, 2, \dots, T\}$$

- Returns of the asset for T time periods

$$R_t = \log(P_t/P_{t-1}), t = 1, 2, \dots, T$$

- $\{R_t\}$ assumed covariance stationary with

$$\sigma = \sqrt{\text{var}(R_t)} = \sqrt{E[(R_t - E[R_t])^2]}$$

with sample estimate:

$$\hat{\sigma} = \sqrt{\frac{1}{T-1} \sum_{t=1}^T (R_t - \bar{R})^2}, \text{ with } \bar{R} = \frac{1}{T} \sum_{t=1}^T R_t.$$

- Annualized values

$$\widehat{\text{vol}} = \begin{cases} \sqrt{252} \hat{\sigma} & \text{(daily prices for 252 business days/year)} \\ \sqrt{52} \hat{\sigma} & \text{(weekly prices)} \\ \sqrt{12} \hat{\sigma} & \text{(monthly prices)} \end{cases}$$

Prediction Methods Based on Historical Volatility

Definition For time period t , define the **sample volatility**

$\hat{\sigma}_t$ = sample standard deviation of period t returns

- If t indexes months with daily data, then $\hat{\sigma}_t$ is the sample standard deviation of daily returns in month t .
- If t indexes days with daily data, then $\hat{\sigma}_t^2 = R_t^2$.
- With high-frequency data, daily σ_t is derived from cumulating squared intra-day returns.

Historical Average: $\tilde{\sigma}_{t+1}^2 = \frac{1}{t} \sum_1^t \hat{\sigma}_j^2$

(uses all available data)

Simple Moving Average: $\tilde{\sigma}_{t+1}^2 = \frac{1}{m} \sum_0^{m-1} \hat{\sigma}_{t-j}^2$

(uses last m single-period sample estimates)

Exponential Moving Average: $\tilde{\sigma}_{t+1}^2 = (1 - \beta)\hat{\sigma}_t^2 + \beta\tilde{\sigma}_t^2$ $0 \leq \beta \leq 1$

(uses all available data)

Exponential Weighted Moving Average:

$$\tilde{\sigma}_{t+1}^2 = \sum_{j=0}^{m-1} (\beta^j \hat{\sigma}_{t-j}^2) / [\sum_{j=0}^{m-1} \beta^j] \text{ (uses last } m \text{)}$$

Predictions Based on Historical Volatility

Simple Regression:

$$\tilde{\sigma}_{t+1}^2 = \gamma_{1,t} \hat{\sigma}_t^2 + \gamma_{1,t} \hat{\sigma}_{t-1}^2 + \cdots + \gamma_{p,t} \hat{\sigma}_{t-p+1}^2 + u_t$$

Regression can be fit using all data or last m (rolling-windows).

Note: similar but different from auto-regression model of $\hat{\sigma}_t^2$

Trade-Offs

- Use more data to increase precision of estimators
- Use data closer to time t for estimation of σ_t .

Evaluate out-of-sample performance

- Distinguish assets and asset-classes
- Consider different sampling frequencies and different forecast horizons
- Apply performance measures (MSE, MAE, MAPE, etc.)

Benchmark Methodology: RiskMetrics

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Geometric Brownian Motion (GBM)

For $\{S(t)\}$ the price of a security/portfolio at time t :

$$dS(t) = \mu S(t)dt + \sigma S(t)dW(t),$$

where

- σ is the volatility of the security's price
- μ is mean return (per unit time).
- $dS(t)$ infinitesimal increment in price
- $dW(t)$ infinitesimal increment of a standard Brownian Motion/Wiener Process
 - Increments $[W(t') - W(t)]$ are Gaussian with mean zero and variance $(t' - t)$.
 - Increments on disjoint time intervals are independent.

For $t_1 < t_2 < t_3 < t_4$,

$[W(t_2) - W(t_1)]$ and $[W(t_4) - W(t_3)]$ are independent

Geometric Brownian Motion (GBM)

Sample Data from Process:

- Prices: $\{S(t), t = t_0, t_1, \dots, t_n\}$
- Returns: $\{R_j = \log[S(t_j)/S(t_{j-1})], j = 1, 2, \dots, n\}$
 indep. r.v.'s: $R_j \sim N(\mu_* \Delta_j, \sigma^2 \Delta_j)$, where
 $\Delta_j = (t_j - t_{j-1})$ and $\mu_* = [\mu - \sigma^2/2]$

($\{\log[S(t)]\}$ is Brownian Motion with drift μ and volatility σ^2 .)

Maximum-Likelihood Parameter Estimation

- If $\Delta_j \equiv 1$, then

$$\begin{aligned}\hat{\mu}_* &= \bar{R} = \frac{1}{n} \sum_{t=1}^n R_t \\ \hat{\sigma}^2 &= \frac{1}{n} \sum_{t=1}^n (R_t - \bar{R})^2\end{aligned}$$

- If Δ_j varies ... Exercise.

Geometric Brownian Motion

Garman-Klass Estimator:

- Sample information more than period-close prices, also have period-high, period-low, and period-open prices.
- Assume $\mu = 0$, $\Delta_j \equiv 1$ (e.g., daily) and let $f \in (0, 1)$ denote the fraction of the day prior to the market open.

$$C_j = \log[S(t_j)]$$

$$O_j = \log[S(t_{j-1} + f)]$$

$$H_j = \max_{t_{j-1}+f \leq t \leq t_j} \log[S(t)]$$

$$L_j = \min_{t_{j-1}+f \leq t \leq t_j} \log[S(t)]$$

Garman-Klass Estimator

Using data from the first period:

- $\hat{\sigma}_0^2 = (C_1 - C_0)^2$: Close-to-Close squared return
 $E[\hat{\sigma}_0^2] = \sigma^2$, and $var[\hat{\sigma}_0^2] = 2(\sigma^2)^2 = 2\sigma^4$.
- $\hat{\sigma}_1^2 = \frac{(O_1 - C_0)^2}{f}$: Close-to-Open squared return
 $E[\hat{\sigma}_1^2] = \sigma^2$, and $var[\hat{\sigma}_1^2] = 2(\sigma^2)^2 = 2\sigma^4$.
- $\hat{\sigma}_2^2 = \frac{(C_1 - O_1)^2}{1-f}$: Open-to-Close squared return
 $E[\hat{\sigma}_2^2] = \sigma^2$, and $var[\hat{\sigma}_2^2] = 2(\sigma^2)^2 = 2\sigma^4$.

Note: $\hat{\sigma}_1^2$ and $\hat{\sigma}_2^2$ are independent!

- $\hat{\sigma}_*^2 = \frac{1}{2}\hat{\sigma}_1^2 + \frac{1}{2}\hat{\sigma}_2^2$
 $E[\hat{\sigma}_*^2] = \sigma^2$, and $var[\hat{\sigma}_*^2] = \sigma^4$.

$$\implies \text{eff}(\hat{\sigma}_*^2) = \frac{\text{var}(\hat{\sigma}_0^2)}{\text{var}(\hat{\sigma}_*^2)} = 2.$$

Parkinson (1976): With $f = 0$, defines

$$\hat{\sigma}_3^2 = \frac{(H_1 - L_1)^2}{4(\log 2)} \text{ and shows } \text{eff}(\hat{\sigma}_3^2) \approx 5.2.$$

Garman and Klass (1980) show that for any $0 < f < 1$:

- $\hat{\sigma}_4^2 = a \times \hat{\sigma}_1^2 + (1 - a)\hat{\sigma}_3^2$

has minimum variance when $a \approx 0.17$, independent of f and
 $\text{Eff}(\hat{\sigma}_4^2) \approx 6.2.$

- “Best Analytic Scale-Invariant Estimator”

$$\hat{\sigma}_{**}^2 = 0.511(u_1 - d_1)^2 - 0.019\{c_1(u_1 + d_1) - 2u_1d_1\} - 0.383c_1^2,$$

where the normalized high/low/close are:

$$u_j = H_j - O_j$$

$$d_j = L_j - O_j$$

$$c_j = C_j - O_j$$

and $\text{Eff}(\hat{\sigma}_{**}^2) \approx 7.4$

- If $0 < f < 1$ then the opening price O_1 may differ from C_0 and the composite estimator is

$$\hat{\sigma}_{GK}^2 = a \frac{(O_1 - C_0)^2}{f} + (1 - a) \frac{\sigma_{**}^2}{(1 - f)}$$

which has minimum variance when $a = 0.12$ and

$$Eff(\hat{\sigma}_{GK}^2) \approx 8.4.$$

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Poisson Jump Diffusions

For $\{S(t)\}$ the stochastic process for the price of the security/portfolio at time t ,

$$\frac{dS(t)}{S(t)} = \mu dt + \sigma dW(t) + \gamma \sigma Z(t) d\Pi(t),$$

where

- $dS(t)$ = infinitesimal increment in price.
- μ = mean return (per unit time)
- σ = diffusion volatility of the security's price process
- $dW(t)$ = increment of standard Wiener Process
- $d\Pi(t)$ = increment of a Poisson Process with rate λ , modeling the jump process.
- $(\gamma\sigma) \times Z(t)$, the magnitude a return jump/shock
 $Z(t)$ i.i.d $N(0, 1)$ r.v.'s and
 γ = scale(σ units) of jump magnitudes.

Poisson Jump Diffusions

Maximum-Likelihood Estimation of the PJD Model

- Model is a Poisson mixture of Gaussian Distributions.
- Moment-generating function derived as that of random sum of independent random variables.
- Likelihood function product of infinite sums
- EM Algorithm* expressible in closed form
 - Jumps treated as latent variables which simplify computations
 - Algorithm provides a posteriori estimates of number of jumps per time period.

* See Pickard, Kempthorne, Zakaria (1987).

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ARCH Models

ARCH models are specified relative to the discrete-time process for the price of the security/portfolio: $\{S_t, t = 1, 2, \dots\}$

Engle (1982) models the discrete returns of the process

$$y_t = \log(S_t/S_{t-1}) \text{ as}$$

$$y_t = \mu_t + \epsilon_t,$$

where μ_t is the mean return, conditional on \mathcal{F}_{t-1} , the information available through time $(t - 1)$, and

$$\epsilon_t = Z_t \times \sigma_t,$$

where Z_t i.i.d. with $E[Z_t] = 0$, and $\text{var}[Z_t] = 1$,

$$\sigma_t^2 = \alpha_0 + \alpha_1 \epsilon_{t-1}^2 + \alpha_2 \epsilon_{t-2}^2 + \dots + \alpha_p \epsilon_{t-p}^2$$

Parameter Constraints: $\alpha_j \geq 0, j = 0, 1, \dots, p$

$\sigma_t^2 = \text{var}(R_t | \mathcal{F}_{t-1})$, “Conditional Heteroscedasticity” of returns .

ARCH Models

The ARCH model:

$$\sigma_t^2 = \alpha_0 + \alpha_1 \epsilon_{t-1}^2 + \alpha_2 \epsilon_{t-2}^2 + \cdots + \alpha_p \epsilon_{t-p}^2$$

implies an AR model in ϵ_t^2 . Add $(\epsilon_t^2 - \sigma_t^2) = u_t$ to both sides:

$$\epsilon_t^2 = \alpha_0 + \alpha_1 \epsilon_{t-1}^2 + \alpha_2 \epsilon_{t-2}^2 + \cdots + \alpha_p \epsilon_{t-p}^2 + u_t$$

where $u_t : E[u_t | \mathcal{F}_t] = 0$, and $\text{var}[u_t | \mathcal{F}_t] = \text{var}(\epsilon_t^2) = 2\sigma_t^4$.

Lagrange Multiplier Test

$$H_0 : \alpha_1 = \alpha_2 = \cdots = \alpha_p = 0$$

- Fit linear regression on squared residuals $\hat{\epsilon}_t = y_t - \hat{\mu}_t$.
(i.e., Fit an AR(p) model to $[\hat{\epsilon}_t^2]$, $t = 1, 2, \dots, n$)
- LM test statistic = nR^2 , where R^2 is the R-squared of the fitted AR(p) model.

Under H_0 the r.v. nR^2 is approx. χ^2 ($df = p$)

- Note: the linear regression estimates of parameters are not MLEs under Gaussian assumptions; they correspond to quasi-maximum likelihood estimates (QMLE).

Maximum Likelihood Estimation

ARCH Model:

$$\begin{aligned}
 y_t &= c + \epsilon_t \\
 \epsilon_t &= z_t \sigma_t \\
 \sigma_t^2 &= \alpha_0 + \alpha_1 \epsilon_{t-1}^2 + \cdots + \alpha_p \epsilon_{t-p}^2 \\
 &\quad t = 0, 1, \dots, T
 \end{aligned}$$

Likelihood:

$$\begin{aligned}
 L(c, \alpha) &= p(y_1, \dots, y_n \mid c, \alpha_0, \alpha_1, \dots, \alpha_p) \\
 &= \prod_{t=1}^n p(y_t \mid \mathcal{F}_{t-1}, c, \alpha) \\
 &= \prod_{t=1}^n \left[\frac{1}{\sqrt{2\pi\sigma_t^2}} \exp\left(-\frac{1}{2} \frac{\epsilon_t^2}{\sigma_t^2}\right) \right] \\
 &\quad \text{where } \epsilon_t = y_t - c \text{ and} \\
 &\quad \sigma_t^2 = \alpha_0 + \alpha_1 \epsilon_{t-1}^2 + \cdots + \alpha_p \epsilon_{t-p}^2.
 \end{aligned}$$

Constraints:

- $\alpha_i \geq 0, i = 1, 2, \dots, p$
- $(\alpha_1 + \cdots + \alpha_p) < 1.$

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GARCH Models

Bollerslev (1986) extended ARCH models to:

GARCH(p,q) Model

$$\sigma_t^2 = \alpha_0 + \sum_{i=1}^p \alpha_i \epsilon_{t-i}^2 + \sum_{j=1}^q \beta_j \sigma_{t-j}^2$$

Constraints: $\alpha_i \geq 0, \forall i$, and $\beta_j \geq 0, \forall j$

GARCH(1,1) Model

$$\sigma_t^2 = \alpha_0 + \alpha_1 \epsilon_{t-1}^2 + \beta_1 \sigma_{t-1}^2$$

- Parsimonious
- Fits many financial time series

GARCH Models

The GARCH(1,1) model:

$$\sigma_t^2 = \alpha_0 + \alpha_1 \epsilon_{t-1}^2 + \beta_1 \sigma_{t-1}^2$$

implies an ARMA model in ϵ_t^2 . Eliminate $\sigma_{t'}^2$ using $(\epsilon_{t'}^2 - \sigma_{t'}^2) = u_{t'}$

$$\epsilon_t^2 - u_t = \alpha_0 + \alpha_1 + \epsilon_{t-1}^2 + \beta_1(\epsilon_{t-1}^2 - u_{t-1})$$

$$\epsilon_t^2 = \alpha_0 + (\alpha_1 + \beta_1)\epsilon_{t-1}^2 + u_t - \beta_1 u_{t-1}$$

where $u_t : E[u_t | \mathcal{F}_t] = 0$, and $\text{var}[u_t | \mathcal{F}_t] = \text{var}(\epsilon_t^2) = 2\sigma_t^4$.

\implies GARCH(1,1) implies an ARMA(1,1) with

$$u_t = (\epsilon_t^2 - \sigma_t^2) \sim WN(0, 2\sigma_t^4)$$

Stationarity of GARCH model deduced from ARMA model

$$A(L)\epsilon_t^2 = B(L)u_t$$

$$\epsilon_t^2 = [A(L)]^{-1}B(L)u_t.$$

Covariance stationary: roots of $A(z)$ outside $\{|z| \leq 1\}$, i.e.,

$$|\alpha_1 + \beta_1| < 1$$

GARCH Models

Unconditional Volatility / Long-Run Variance

GARCH(1,1): Assuming stationarity, $0 < (\alpha_1 + \beta_1) < 1$

$$\sigma_*^2 = \alpha_0 + (\alpha_1 + \beta_1)\sigma_*^2$$

$$\implies \sigma_*^2 = \frac{\alpha_0}{(1 - \alpha_1 - \beta_1)}$$

GARCH(p,q) implies ARMA(max(p,q), q) model

- Stationary if $0 < (\sum_1^p \alpha_i + \sum_1^q \beta_j) < 1$
- Long-Run Variance:

$$\sigma_*^2 = \alpha_0 + (\sum_1^p \alpha_j + \sum_1^q \beta_1)\sigma_*^2$$

$$\implies \sigma_*^2 = \frac{\alpha_0}{[1 - \sum_1^{\max(p,q)} (\alpha_i + \beta_j)]}$$

GARCH Model Estimation

Maximum Likelihood Estimation

GARCH Model:

$$y_t = c + \epsilon_t$$

$$\epsilon_t = z_t \sigma_t$$

$$\sigma_t^2 = \alpha_0 + \sum_{i=1}^p \alpha_i \epsilon_{t-i}^2 + \sum_{j=1}^q \beta_j \sigma_{t-j}^2$$

$t = 0, 1, \dots, T$

Likelihood:

$$\begin{aligned} L(c, \alpha, \beta) &= p(y_1, \dots, y_T \mid c, \alpha_0, \alpha_1, \dots, \alpha_p, \beta_1, \dots, \beta_q) \\ &= \prod_{t=1}^T p(y_t \mid \mathcal{F}_{t-1}, c, \alpha, \beta) \\ &= \prod_{t=1}^T \left[\frac{1}{\sqrt{2\pi\sigma_t^2}} \exp\left(-\frac{1}{2} \frac{\epsilon_t^2}{\sigma_t^2}\right) \right] \end{aligned}$$

where $\epsilon_t = y_t - c$ and

$$\sigma_t^2 = \alpha_0 + \sum_{i=1}^p \alpha_i \epsilon_{t-i}^2 + \sum_{j=1}^q \beta_j \sigma_{t-j}^2.$$

Constraints: $\alpha_i \geq 0, \forall i$, $\beta_j \geq 0, \forall j$, and $0 < (\sum_{i=1}^p \alpha_i + \sum_{j=1}^q \beta_j) < 1$.

GARCH Model Estimation/Evaluation/Model-Selection

- Maximum-Likelihood Estimates: $\hat{\alpha}, \hat{\beta}$
 $\implies \hat{\epsilon}_t$ and $\hat{\sigma}_t^2$ ($t = T, T - 1, \dots$)
- Standardized Residuals
 $\hat{\epsilon}_t / \hat{\sigma}_t$: should be uncorrelated
- Squared Standardized Residuals
 $(\hat{\epsilon}_t / \hat{\sigma}_t)^2$: should be uncorrelated

Testing Normality of Residuals

- Normal QQ Plots
- Jaque-Bera test
- Shapiro-Wilk test
- MLE Percentiles Goodness-of-Fit Test
- Kolmogorov-Smirnov Goodness-of-Fit Test

Model Selection: Apply model-selection criteria

- Akaike Information Criterion (AIC)
- Bayes Information Criterion (BIC)

Stylized Features of Returns/Volatility

Volatility Clustering

- Large ϵ_t^2 follow large ϵ_{t-1}^2
- Small ϵ_t^2 follow small ϵ_{t-1}^2

GARCH models can prescribe

- Large σ_t^2 follow large σ_{t-1}^2
- Small σ_t^2 follow small σ_{t-1}^2

Heavy Tails / Fat Tails

- Returns distribution has heavier tails (higher Kurtosis) than Gaussian
- GARCH(p,q) models are stochastic mixture of Gaussian distributions with higher kurtosis.

Engle, Bollerslev, and Nelson (1994)

Stylized Features of Returns/Volatility

Volatility Mean Reversion

GARCH(1,1) Model

- Long-run average volatility: $\sigma_*^2 = \frac{\alpha_0}{1 - \alpha_1 - \beta_1}$
- Mean-Reversion to Long-Run Average

$$\epsilon_t^2 = \alpha_0 + (\alpha_1 + \beta_1)\epsilon_{t-1}^2 + u_t - \beta_1 u_{t-1}$$

Substituting: $\alpha_0 = (1 - \alpha_1 - \beta_1)\sigma_*^2$

$$(\epsilon_t^2 - \sigma_*^2) = (\alpha_1 + \beta_1)(\epsilon_{t-1}^2 - \sigma_*^2) + u_t - \beta_1 u_{t-1}$$

$0 < (\alpha_1 + \beta_1) < 1 \implies$ Mean Reversion!

Extended Ornstein-Uhlenbeck(OU) Process for ϵ_t^2 with MA(1) errors.

Extensions of GARCH Models

EGARCH Nelson (1992)

TGARCH Glosten, Jagannathan, Runkler (1993)

PGARCH Ding, Engle, Granger

GARCH-In-Mean

Non-Gaussian Distributions

- t -Distributions
- Generalized Error Distributions (GED)

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