18.S096 Problem Set 8 Fall 2013
Stochastic Calculus

Due date **: 11/21/13**
Collaboration on homework is encouraged, but you should think through the problems yourself before discussing them with other people. **You must write your solution in your own words. Make sure to list all your collaborators.**

Part A

Part A has problems that straightforwardly follow from the definition. Use this part as an opportunity to get used to the concepts and definitions.

Problem A-1. Identify Martingales.
(a) Simple random walk
(b) The process $X_t = |S_t|$, where S_t is a simple random walk.
(c) $X_0 = 0$ and $X_{t+1} = X_t + (Y_t - \frac{1}{\lambda})$ for $t \geq 0$, where Y_t are i.i.d. random variables with exponential distribution of parameter λ.
(d) $X_0 = 0$ and $X_{t+1} = X_t + Z_t Z_{t-1} \cdot Z_0$ for $t \geq 0$, where Z_t are i.i.d. random variables with log-normal distribution.
(e) $X_t = B_t^{(1)} B_t^{(2)}$ where $B_t^{(1)}$ and $B_t^{(2)}$ are independent Brownian motions.

Problem A-2. (a) Let $B(t)$ be a Brownian motion. Compute

$$\mathbb{E}[B(t) \,|\, B(s)] \qquad \text{and} \qquad \mathbb{V}[B(t) \,|\, B(s)],$$

where $t > s \geq 0$ are fixed reals.
(b) Let $X(t)$ be a Brownian motion with drift μ. Compute

$$\mathbb{E}[X(t) \,|\, X(s)] \qquad \text{and} \qquad \mathbb{V}[X(t) \,|\, X(s)],$$

where $t > s \geq 0$ are fixed reals.
(c) For a Brownian motion B and two fixed reals t and s satisfying $t > s$, compute

$$\mathbb{E}[exp(\sigma(B(t) - B(s))].$$

Problem A-3. Let X_t be a given stochastic process. Identify processes adapted to X_t among the following stochastic processes Y_t,.
(a) $Y_t = X_{T-t}$ for $t \leq T$, for some fixed T.
(b) $Y_t = \max_{0 \leq x \leq 2t} X_s$.
(c) $Y_t = |\{i \in [0, t] \,:\, X_i \geq 0\}|$.

Problem A-4. Use Ito's formula to compute the differentials of the following functions (B_t is a Brownian motion):
(a) $f(t, B_t) = B_t^3$.
(b) $f(t, B_t) = \sin B_t$.

(c) $f(t, B_t) = \cos(t^3 + B_t^2)$.
(d) $f(t, B_t) = e^{B_t^2}$.
(e) $f(t, B_t) = \int B_t^2 dB_t$.
(f) $f(t, B_t) = \int B_t dt$
(g) $f(t, X_t) = X_t^2$, where $dX_t = \mu dt + \sigma dB_t$ (μ and σ are constants).

Problem A-5. Let $B(t)$ be a Brownian motion. Prove that $B(t)$ and $B(t)^2$ are not equivalent probability distributions (two probability distributions $\mathbf{P}$ and $\mathbf{Q}$ are equivalent if for all sets X, $\mathbf{P}(X) > 0$ if and only if $\mathbf{Q}(X) > 0$).

Part B

Part B has more elaborate problems. Many of the problems in Part B cover important topics that we did not have enough time to cover in lecture. Thus understanding the content is as important as solving the problem. Try to think through the content of the problem while solving it.

Problem B-1. Let $Y(t) = (X_1(t), X_2(t))$, where $X_1(t)$ and $X_2(t)$ are independent random Brownian motions (thus $Y(t)$ is a 2-dimensional Brownian motion).
(i) For a fixed value of t, find the probability density function of $Y(t)$.
(ii) Let $D_\rho = \{x \in \mathbb{R}^2 : |x| < \rho\}$, and compute $\mathbf{P}(Y(t) \in D_\rho)$.

Problem B-2. Use Ito isometry to caculate $\mathbb{E}X_t^2$, where

$$X_t = \int_0^t B_s dB_s.$$

Problem B-3. (i) Prove the following integration by parts formula for Ito integral:

$$\int_0^t h(s) dB_s = h(t)B_t - \int_0^t h'(s) B_s ds.$$

(ii) By using part (i), prove that for each fixed T, the random variable

$$\int_0^T B_s ds$$

has normal distribution.