Simplicity Theory

3/3. Josh's Examples Talk.

① κ-saturated κ-strongly homogeneous structure $\mathcal{U}$, $\Delta = \mathcal{L}_{\omega,\omega}$
$\mathcal{U}$ is strongly hom. by assumption.
Compactness: if Σ is a set of formulas finitely realisable in $\mathcal{U}$ $\Rightarrow$ realisable in $\mathcal{U}$.

Lemma Suppose Δ is the set of all positive qf formulas on predicates $\{R_i\}$, and suppose $\mathcal{U}$ has compactness for sets of predicates. Then it has compactness for subsets of Δ.

Proof Let $\Sigma \subset \Delta$ be a set of formulas that is finitely realisable. Let $\bar{\Sigma}$ be a maximal finitely realisable subset containing Σ, (with same free variables).

If $\varphi \vee \psi \in \bar{\Sigma}$, we claim $\varphi \in \bar{\Sigma}$ or $\psi \in \bar{\Sigma}$. Otherwise we can find $\varphi', \psi' \in \bar{\Sigma}$ s.t. $\varphi \wedge \varphi'$ not realisable, $\psi' \wedge \psi$ not realisable, but then $\varphi' \wedge \psi' \wedge (\varphi \vee \psi)$ is not realisable ※.

So put each $\varphi \in \bar{\Sigma}$ in disjunctive normal form $\varphi = \underline{\varphi_1} \vee \dots \vee \varphi_n$
$\Rightarrow$ one of $\underline{\varphi_i} \in \bar{\Sigma}$ $\Rightarrow$ every conjunct of $\varphi_i \in \bar{\Sigma}$.
$\Rightarrow$ $\{R_i\} \cap \bar{\Sigma} \vdash \bar{\Sigma} \vdash \Sigma$
finitely realisable $\Rightarrow$ realisable $\Rightarrow$ realisable $\square$.

② Hilbert Space Example.

$\mathcal{U}$ = unit closed ball in a large Hilbert space $\mathcal{H}$.
$\mathcal{L} = \{ \| \Sigma \lambda_i x_i \| \substack{\geq r \\ \leq r}, \lambda_i \in \mathbb{R}, r \in \mathbb{R} \}$
Δ = positive qf formulas in $\mathcal{L}$.

(Rmk: inner product is expressible in these terms.
$(x, y) = \dfrac{\|x+y\|^2 - \|x-y\|^2}{4}$).

$$(x,y) \leq r \iff \|x+y\|^2 \leq \|x-y\|^2 + 4r$$
$$\iff \bigwedge_{n \in \omega} \bigvee_{k=0}^{4n} \left(\frac{k}{n} \leq \|x-y\|^2 \right) \wedge \|x+y\|^2 \leq \frac{k+1}{n} + 4r$$

Check $\mathcal{U}$ is a universal domain:

Homogeneous: Let $f : A \to \mathcal{U}$ be a partial homomorphism. So for $x_i \in A$, $\lambda_i \in \mathbb{R}$, we know

$$\sum \lambda_i x_i = 0 \iff \| \sum \lambda_i x_i \| = 0 \Rightarrow \| \sum \lambda_i f(x_i) \| = 0 \Rightarrow \sum \lambda_i f(x_i) = 0.$$

So f extends ~~an automorphi~~ to $f : \langle A \rangle \to \mathcal{H}$.
So since ~~(x,y) is~~ $(,)$ is type definable, it extends to $\overline{\langle A \rangle} \to \mathcal{H}$.
f preserves the metric so this is an embedding.

→ Hilbert dim > κ.
So since $\mathcal{H}$ is large, we can pick an isomorphism $\langle \overline{A} \rangle^{\perp} \to f(\langle \overline{A} \rangle)^{\perp}$, and we get an onto $f : \mathcal{H} \to \mathcal{H}$ restricting to $\mathcal{U} \to \mathcal{U}$.

Compactness: Start with $\Sigma \subseteq \mathcal{L}$, let $\{x_i\}$ be the variables in Σ.
Let $W = \bigoplus \mathbb{R} x_i$, let $W_0 \subseteq W$ st. $W_0 = \{ \sum \lambda_i x_i \mid \sum \lambda_i^2 \leq 1 \}$.

Look at $[0,1]^{W_0}$ functions from W_0 to $[0,1]$ with Tickinoff ?
By the lemma, it's enough to consider sets of predicates.
Each predicate $\| \frac{\sum \lambda_i x_i}{\sqrt{\sum \lambda_i^2}} \| \geq \frac{r}{\sqrt{\sum \lambda_i^2}}$ defines a closed subset of $[0,1]^{W_0}$.

The axioms for a semi norm $\|x+y\| \leq \|x\| + \|y\|$ & $\|rx\| = |r| \|x\|$ define a closed subset.

The requirement that $\| \cdot \|$ defines a semi-positive-definite inner product defines a further closed subset. Call it $D \subseteq [0,1]^{W_0}$

$\Sigma = \{ P_j(\overline{x}_j) ; j < \alpha \}$ $\overset{\rightarrow}{\text{defines}}$ closed subsets $C_j \subset [0,1]^{W_0}$.

Furthermore $\{C_j\} \cup \{D\}$ has finite intersection property
since compact
$\Rightarrow \bigcap C_j \cap D \neq \phi$. Let $\widetilde{\| \cdot \|} \in \bigcap C_j \cap D$. → mod out by elts of norm 0.

So we get a semi-norm on $W \Rightarrow W \twoheadrightarrow V$, $\widetilde{\| \cdot \|}$ descends to $\| \cdot \|$.

$\Rightarrow V \hookrightarrow \mathcal{H}$

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$\Rightarrow x_i \longmapsto a_i \in \mathcal{U} \subseteq \mathcal{H}. \Rightarrow a_i$ realise Σ.
↑variables

③ Hyperimaginaries.

Let $\mathcal{U}, \mathcal{L}, \Delta$ be a universal domain, let $\alpha < \kappa$ be an ordinal, and E is a type-definable equivalence reln on $\mathcal{U}^\alpha$.

Let $\mathcal{U}' = \mathcal{U} \amalg \underset{\alpha\text{-tuples}}{\mathcal{U}^\alpha}/E$. Let $\mathcal{L}' = \{ \varphi_E(\underbrace{x_0, x_1, x_2, \ldots}_{\text{in home sort}}, \underbrace{(y_0)_E, (y_1)_E, \ldots}_{\text{imaginary sort}})$

st. $\varphi(x_0, x_1, \ldots, \overset{\rightarrow}{\overline{y_0}}, \overline{y_1}, \ldots) \in \Delta \}$.
↑lots of dummy variables in here.

Interpretation: $\varphi_E(a_0 a_1 \ldots (b_0)_E (b_1)_E \ldots) \Leftrightarrow \exists \overline{b_i}' \in (b_i)_E$ st. $\varphi(a_0, a_1, \ldots, \overline{b_i'}, \overline{b_1}, \ldots)$

$\Delta' =$ positive Vf formulas

Lemma TFAE: For a & b $\overset{\in \mathcal{U}}{}$ fixed. → can prove for any tuples of mixed sorts.

(i) $tp(a_E) = tp(b_E)$

(ii) $\exists c \in a_E$ st. $c \equiv b$

(iii) $\exists c \in a_E, d \in b_E, c \equiv d$.

(i) ⇒ (ii) Enough to show that $\forall \varphi \in tp(b) \models E(x,a) \wedge \varphi(x)$.

But $\varphi_E(x_E) \in tp(b_E) = tp(a_E)$, $\exists c \in a_E$ st. $\varphi(c)$
ie $\models E(x,a) \wedge \varphi(x)$.

(ii) ⇒ (iii) clear.

(iii) ⇒ (i) Enough to show for each $\varphi_E \in tp(a_E)$, $\varphi_E(b_E)$.

By homogeneity $\exists$ aut $f : c \mapsto d$ so $E(f(a), d)$. whence $E(f(a), b)$.

Let $e \models E(x,a) \wedge \varphi$. Then $E(f(a), f(a)) \Rightarrow E(f(e), b)$.

But $\varphi(f(e))$ so $f(e) \models E(x,b) \wedge \varphi$ $\quad\Box$.

<u>Homogeneity of $\mathcal{U}'$</u>: Start with $f : A \to \mathcal{U}'$ partial homomorphism.
~~$f : A_n \sqcup A_i \to \mathcal{U} \sqcup \mathcal{U}^*/E$.~~

Same proof as (i) $\Rightarrow$ (ii) above shows that

$$tp_\Delta(a_0, a_1, \ldots, (b_0)_E, (b_1)_E, \ldots) \subseteq tp_\Delta(c_0, c_1, \ldots, (d_0)_E, (d_1)_E, \ldots)$$

$\Rightarrow$ $\exists\, e_0, e_1, \ldots$ st. $E(e_i, d_i)$ st. $tp_\Delta(a_0\, a_1 \cdots b_0\, b_1 \cdots) \subseteq$

$$tp_\Delta(c_0, c_1, \ldots, e_0, e_1, \ldots).$$

Now use homogeneity in $\mathcal{U}$. ie map sending $a_i \mapsto c_i$ & $b_i \mapsto e_i$ extends to ~~an~~ an automorphism of $\mathcal{U}$.

This extends uniquely to $\mathcal{U}'$.

Compactness of $\mathcal{U}'$: From previous lemma, it's enough to check it on sets of predicates.

Suppose $\{(\varphi_i)_E\}$ is a set of predicates, finitely realisable in $\mathcal{U}'$. Then
$$\bar{\Sigma} = \{ \varphi_i(x_0, x_1, \dots, z_0^i, z_1^i \dots) \wedge \bigwedge_j E(y_j, z_j^i) \}$$
is finitely realisable in $\mathcal{U}$.
$\Rightarrow$ realisable in $\mathcal{U}$ by $x_k = a_k$, $z_j^i := c_j^i$, $y_j := b_j$,
whence Σ is realisable in $\mathcal{U}'$ by a_k, $(b_j)_E$. $\square$

Back to Hilbert Space example...

$\mathcal{U}$ is unit ball in large Hilbert space $\mathcal{H}$, $\Delta =$ positive qf formulas on predicates $\{ \|\sum \lambda_i x_i\| \underset{\leq r}{\geq r} \}$.

Let $A \underset{C}{\perp} B$ mean that $\underset{\text{projection onto cl of } A}{P_C(A)} = P_{CB}(A)$. ($P_D$ is just projection onto $\overline{\langle D \rangle}$)

Claim: $\perp = \mathop{\smile\mkern-11mu|}$ ie $\perp$ is a simple notion of a independence relation.

① invariance under automorphisms
respects norm, so respects ~~inp~~ inner products so respects $\perp$.

Remark $A \underset{C}{\perp} B \iff P_L(A) \perp P_L(B)$ where $L = \overline{\langle C \rangle}^{\perp}$

$A \underset{C}{\perp} B \iff P_C(A) = P_{CB}(A) \iff P_{CB}(A) \subseteq \overline{\langle C \rangle}$

$\iff P_L \; P_{CB}(A) = 0 \iff P_{P_L(B)}(A) = 0 \iff P_{P_L(B)}(P_C(A))^{=0}$,

$\iff P_L(A) \perp P_L(B)$

② Finite character. use $P_L(A) \perp P_L(B)$ & finiteness.

③ Symmetry: obvious. (may need something ↓).

④ Transitivity: Let $L' = \overline{\langle B \rangle}^{\perp}$, $A \underset{C}{\perp} BD \Leftrightarrow P_L(A) \perp P_L(BD)$

$\Leftrightarrow P_L(A) \perp P_L(B)$ & $P_L(A) \perp P_L(D)$

d be But $P_L(A) \perp P_L(B) \Rightarrow P_L(A) \subseteq L'$, so $\underline{P_{L'}(A)} = P_L(A) \perp P_L(D)$,

npler but $P_L(D) = P_{L'}(D)$ + something in $\langle P_L(B) \rangle$

$\Rightarrow P_{L'}(A) \perp P_{L'}(D)$.

ie $A \underset{C}{\perp} BD \Leftrightarrow P_C(A) = P_{BCD}(A) = P_{BC}(A) \Leftrightarrow A \underset{C}{\perp} B$
$\& \; A \underset{BC}{\perp} D$

⑤ Extension: Given A, B, C. Let $L = \overline{\langle C \rangle}^{\perp}$.
Let f be an automorphism of $\mathcal{H}$ fixing C & sending $P_L(A)$ into the orth complement of $P_L(B)$ in L.
Then $A' = f(A)$ has the desired property (since $P_L(A') = P_{fL}(f(A)) = f(P_L(A)) \perp P_L(B)$).

⑥ Local character: Let A be finite, & B arbitrary.
Looking for $B' \subseteq B$ with $|B'| \leq \omega$ so that $A \underset{B'}{\perp} B$,

ie $P_{B'}(A) = P_B(A)$. For each $\vec{a} \in P_B(A)$, let b_i be a sequence in the finite span of B converging to A.

Let $B_a = \{$all vectors appearing in some $b_i\}$. Then $\bigcup_{a \in P_B(A)} B_a = B'$ is what we want.

⑦ Independence Thm:

Lemma: Every $tp(A/C)$ has a unique orthogonal extn to a type over CB.

<u>Proof</u>: existence we have by extension, so enough to prove uniqueness.

So suppose we have $A_1 \models tp(A/C)$ s.t. $A_1 \underset{C}{\perp} B$.

Then we have a C-automorphism sending ~~A_1 to A_2,~~ (A to ~~$P_L(A)$~~ A_1,) ~~is sending $\langle A_1 \rangle$~~ $P_L(A)$ into orthog complement of $P_L(B)$ in L.

Claim this determines $tp(A_1/BC)$, because it determines the norm on $\langle A_1 \rangle + \langle B \rangle + \langle C \rangle =: V$.

(Suppose we have $v \in V$. ~~Then dist (v, L) is determined by $tp(A_1/C) = tp(A/C)$~~

$v = a + b + c$ where $a \in \langle A_1 \rangle$, $b \in \langle B \rangle$ & $c \in \langle C \rangle$.

Write $a = a' + \underset{\text{"}a\text{"}}{P_C(a)}$ & $b = b' + \underset{\text{"}b\text{"}}{P_C(b)}$.

Then $\|v\|^2 = \|a'\|^2 + \|b'\|^2 + \|a'' + b'' + c\|^2$.

& can calculate these by knowing norm $tp(A/C)$ & $tp(B/C)$ resp

& $P_L(A) \perp P_L(B)$ (or something like it gives last step)

$P_L(C) \leftarrow$ maybe C. Humpf.

$\boxed{\checkmark}$.

Proof of ind thm: Assuming $A_1 \underset{C}{\equiv} A_2$, $B_1 \underset{C}{\perp} B_2$ & $A_i \underset{C}{\perp} B_i$

$\Rightarrow \exists A \underset{C}{\equiv} A_i$, $A \underset{C}{\perp} B_1 B_2$, $A \underset{CB_i}{\equiv} A_i$.

(Note: this is stronger statement than in general for ind thm: types instead of strong types)

Let $A \models tp(A_1/C)$ s.t. $A \underset{C}{\perp} B_1 B_2$. Then from previous lemma,

$A \underset{CB_i}{\equiv} A_i$ $\forall i$ (which was what we wanted...) :)

Note: we didn't need $B_1 \underset{C}{\perp} B_2$. $\square$

End of Josh's talk...