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2/18 $p(x,b)$ divides $/c$ if $\exists$ c-indiscernible sequence (b_i) in $tp(b/c)$ st. $\bigwedge p(x,b_i)$ is inconsistent.

$\Rightarrow \exists\ k<\omega$ & $\varphi(x,b) \in p(x,b)$ st. $\bigwedge_{i<k} \varphi(x,b_i)$ is inconsistent $\Rightarrow \{\varphi(x,b_i)\}$ is k-inconsistent.

If we have negations then we can say $\bigwedge_{i_0<\dots<i_{k-1}} \neg\exists x \bigwedge_{j<k} \varphi(x,b_{i_j})$ and apply ext/ext to get (b_i) indiscernible.

Defn. Let $\varphi(x,y)$ be a formula (x, y tuples of variables), $k<\omega$.

$\psi(y_0 \dots y_{k-1})$ another formula st each y_i has the same length as y. [each y_i is in the sort of y].

Then ψ is a <u>k-inconsistency witness</u> for φ if

$$T \vdash \neg\exists x\bar{y}\ \psi(\bar{y}) \wedge \bigwedge_{i<k} \varphi(x,y_i).$$

Defn: A formula $\varphi(x,b)$ divides $/c$ w.r.t. a k-inconsistency witness $\psi(\bar{y})$ if there exists a sequence (b_i) in $tp(b/c)$ satisfying:

$$\underbrace{\bigwedge_{i_0 \dots i_{k-1}}}_{\text{for all } i_0 \dots i_{k-1}} \psi(b_{i_0} \dots b_{i_{k-1}}) = \tilde{\psi}(\bar{b}) \quad \left[\tilde{\psi}(y_0 \dots) := \bigwedge_{i_0 \dots i_{k-1}} \psi(y_{i_0} \dots y_{i_{k-1}})\right]$$

Prop: ① $\varphi(x,b)$ divides $/c$ $\Leftrightarrow$ ② divides $/c$ wrt some k-inconsistency witness ψ $\Leftrightarrow$ ③ $\exists$ c-indiscernible sequence (b_i) in $tp(b/c)$ s.t. $\psi(b_0 \dots b_{k-1})$.

Proof ① $\Rightarrow$ ② $\exists$ an indiscernible seq (b_i) in $tp(b/c)$ s.t. $\bigwedge p(x,b_i)$ is inconsistent.

By compactness $\exists k < \omega$ s.t. $\bigwedge_{i<k} \varphi(x,b_i)$ is inconsistent.

~~let $q(y_0 \dots y_{k-1}) = tp(b_{<k})$~~

~~$\Rightarrow q(\bar{y}) \wedge \bigwedge_{i<k} p(x,y_i)$ is inconsistent.~~

$\Rightarrow$ $\exists \psi(\bar{y}) \in tp(b_{<k})$ s.t. $\psi(\bar{y}) \wedge \bigwedge_{i<k} \varphi(x,y_i)$ is inconsistent ✓ □.

② $\Rightarrow$ ③ We have a sequence (b_i) in $tp(b/c)$ satisfying $\tilde{\psi}$. Since compactness applies to ψ (it does not apply to $\neg\exists x \bigwedge \varphi(x,y_i)$) we may apply extension/extraction to get a sequence (b_i') indiscernible $/c$ having same properties.

③ $\Rightarrow$ ① clear. ($\bigwedge \varphi(x,b_i)$ is inconsistent because $\models \psi(b_0 \dots b_{k-1})$)

□

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<u>Defn</u> Let x be a tuple of variables.

~~Then $\Xi(x) = \{ (\varphi(x,y), \psi(y$~~

Then $\Xi(x) = \{ (\varphi(x,y), \psi(y_0 \cdots y_{k-1}) : \varphi(x,y) \in \Delta;\ k<\omega;$

$\psi \in \Delta$ is a k-inconsistency witness for $\varphi \}$.

x is fixed but k & y vary.

<u>Defn</u>: For every partial type $p(x)$ (with parameters) we associate a "rank", written $D(p, \Xi)$ which is a set of sequences in Ξ of ordinal length.

For $\xi \in \Xi^{\alpha}$ we decide whether $\xi \in D(p, \Xi)$ by induction on α:

$\alpha = 0$: $\langle\rangle \in D(p, \Xi)$ iff p is consistent.

α limit: $\xi \in D(p, \Xi)$ iff $\forall \beta < \alpha$ $\xi|_{\beta} \in D(p, \Xi)$.

$\alpha = \beta + 1$: $\xi = \langle \theta, (\varphi(x,y), \psi(\bar{y})) \rangle$ where $\theta \in \Xi^{\beta}$.

Assume p is over b.

Then $\xi \in D(p, \Xi)$ iff $\exists c$ s.t. $\varphi(x,c)$ divides$/b$ wrt ψ and $\theta \in D(p(x) \wedge \varphi(x,c), \Xi)$.

Obvious things: • If $\xi \in D(p, \Xi)$ and $p \vdash q$ then $\xi \in D(q, \Xi)$.

• $D(p, \Xi)$ is closed under subsequences.

Still need to get rid of p / b assumption ...

Remark We prove by induction on α that for $\xi \in \Xi^\alpha$ and ~~[illegible]~~ $p(x,b) \equiv q(x,b')$ that $\xi \in D(p(x,b), \Xi)$ iff $\xi \in D(q(x,b'), \Xi)$

(ie choice of set of parameters b is not important)

Proof: $\alpha = 0$ ✓
α limit ✓.

Let $\alpha = \beta+1$, $\xi = \langle \theta, (\varphi, \psi) \rangle$ and assume $\xi \in D(p, \Xi)$.

$\Rightarrow \exists c$ st. $\varphi(x,c)$ divides $/ b$ wrt. ψ, $\theta \in D(p \wedge \varphi(x,c), \Xi)$

$\Rightarrow \exists$ b-indiscernible sequence (c_i) in $tp(c/b)$ st.

$\psi(c_0 \dots c_{k-1})$ and $\theta \in D(p \wedge \varphi(x,c), \Xi) = D(p \wedge \varphi(x,c_i), \Xi)$

since $c \equiv_b c_i$, $p \wedge \varphi(x,c) \equiv p \wedge \varphi(x,c_i)$.

By extension / extraction there is a bb'-indiscernible sequence (c_i') similar over b to (c_i).

(So $\psi(c_0 \dots c_{k-1}) \Rightarrow \psi(c_0' \dots c_{k-1}')$.)

:/5

$\theta \in D(p \wedge \varphi(x, c_0'), \Xi) = \xi\, D(q \wedge \varphi(x, c_0'), \Xi)$

and $\varphi(x, c_0')$ divides $/ bb'$ w.r.t. ψ and thus $/ b'$

$\Rightarrow \xi \in D(q, \Xi)$. $\square$

<u>Defn</u> T is <u>thick</u> if indiscernibility is typedefinable ie

$\forall$ tuple x $\exists$ partial type $\Theta(x_{<\omega})$ saying precisely that

(x_i) is indiscernible.

<u>Remark</u>: Let ~~a~~ b and $(a_i\ i<\omega)$ be possibly infinite tuples. Then (a_i) is indiscernible $/ b$ iff $\forall$ finite subtuples $b' \subseteq b$ and $a_0' \subseteq a_0$, if $a_i' \subseteq a_i$ are the corresponding subtuples, the sequence $(a_i' b' \cdot i<\omega)$ is indiscernible.

all a_i of same length

It follows that for T to be thick, it suffices that indiscernibility of sequences of finite tuples be (type) definable and we get (type) definability of indiscernibility / something.

<u>Remark</u> A first order theory is thick: $\bigwedge_k \bigwedge_{\varphi(x_0 \ldots x_{k-1})} \bigwedge_{\substack{i_0<\cdots<i_{k-1} \\ j_0<\cdots<j_{k-1}}} \varphi(x_{i_0} \ldots x_{i_{k-1}}) \leftrightarrow \varphi(x_{j_0} \ldots x_{j_{k-1}})$ $\square$

Let $p(x,y)$ be a partial type, x & y possibly infinite tuples.

Assume p is closed under finite conjunction.

Let $q(y) = \{\exists x' \varphi(x', y') \cdot x' \subseteq x, y' \subseteq y \text{ finite} : \varphi(x', y') \in p\}$

Then $q(y) \equiv \exists x\, p(x,y)$. By compactness

$\Leftarrow$ clear $\rightarrow$ compactness, if $\models q(b)$, then $p(x; b)$ is consistent. $\square$

Let $\xi \in \Xi^\alpha$, ie $\xi = ((\varphi_i(x, y_i); \psi_i) : i < \alpha)$.

Define $\mathrm{div}_{b,\xi}(x)$ to be the partial type saying:

There exist $c_i : i < \alpha$ of the ~~right~~ lengths of the corresponding y_i s.t.

(1) $\models \bigwedge \varphi_i(x, c_i)$

(2) For all $i < \alpha$, there exists a $b, c_{>i}$-indiscernible sequence $(c_i^j : j < \omega)$ with $c_i^0 = c_i$ and $\models \psi_i(c_i^0 \cdots c_i^{k_i - 1})$.

Prop: Let $p(x)$ be a partial type over C

Then $\xi \in D(p, \Xi)$ iff $p(x) \wedge \mathrm{div}_{C,\xi}(x)$ is consistent.

23. Proof By induction on α, where $\xi = ((\varphi_i, \psi_i) : i < \alpha)$.

$\alpha = 0$, $\langle\rangle \in D(p, \Xi)$ iff p is consistent

iff $p(x) \wedge \underbrace{\mathrm{div}_{C,\langle\rangle}(x)}_{\text{says nothing}}$ is consistent

$\cdot 12$

α limit: $\Leftarrow$ by ind hyp & def of $D(p, \Xi)$.

$\Rightarrow$: by $\underline{\text{compactness}}$ & ind hyp.
Thickness allows compactness.

$\alpha = \beta + 1$: $\xi = \theta \wedge (\varphi_\beta, \psi_\beta)$:

$\xi \in D(p, \Xi) \Leftrightarrow \overbrace{\exists b_\beta \text{ st. } \varphi_\beta(x, b_\beta) \text{ divides} / c \text{ wrt } \psi_\beta}^{(*)}$ and $\theta \in D(p \wedge \varphi_\beta(x, b_\beta), \Xi)$

$\Leftrightarrow (*)$ and $\exists a \models p \wedge \varphi_\beta(x, b_\beta)$ and ~~there are~~

$a \models \operatorname{div}_{cb_\beta, \theta}$, ie $\exists b_i\ i < \beta$ st.

$\bigwedge_{i<\beta} \varphi_i(a, b_i)$ and $\varphi_i(x, b_i)$ divides$/ c b_\beta, b_{\{j : i<j<\beta\}}$ wrt ψ_i

$\Leftrightarrow \exists a$ st. $p(a) \wedge \bigwedge_{i \leq \beta} \varphi_i(a, b_i) \wedge \varphi_i(x, b_i) \operatorname{div} / c b_{\{j : i<j \leq \beta\}}$ $\forall i \leq \beta$

$\Leftrightarrow p(x) \wedge \operatorname{div}_{c, \xi}(x)$ is consistent $\square$.

$\underline{\text{Theorem}}$ (still assuming thickness?) TFAE

① T is simple (ie $\kappa^0(T) < \infty$).

② $\forall (\varphi, \psi) \in \Xi$, $\exists \ell < \omega$ st. there is no sequence $(b_i : i < \ell)$ where each $\varphi(x, b_i)$ divides $/ b_{<i}$ wrt ψ and $\bigwedge_{i<\ell} \varphi(x, b_i)$ is consistent

③ $\kappa^\circ(T) \leq |T|^+$

④ $\forall p \quad D(p, \Xi) \subseteq \Xi^{<|T|^+}$

Proof ① ⇒ ②:

Assume $\kappa^0(T) < \infty$ but ② is false.

ie there are $(\varphi, \psi) \in \Xi$ st. $\forall \ell < \omega \; \exists (b_i : i < \ell)$

st. $\forall i$ $\varphi(x, b_i)$ divides wrt ψ / $b_{<i}$ and $\bigwedge \varphi(x, b_i)$ is consistent

⇒ by compactness $\exists (b_i : i < \kappa^\circ_r(T))$ st.

$\varphi(x, b_i)$ divides wrt. ψ / $b_{<i}$ $\forall i < \kappa^\circ_r(T)$

and $\bigwedge_{i < \kappa^\circ_r(T)} \varphi(x, b_i)$ consistent.

So let $a \models \bigwedge \varphi(x, b_i)$, then $tp(a / b_{<\kappa^\circ_r(T)})$

contradicts the definition of $\kappa^\circ_r(T)$.

② ⇒ ③ Assume ~~weak~~ ③ is false.

Then we have singleton a & set A st.

$tp(a/A)$ divides over every $A_0 \subseteq A$ st. $|A_0| \leq |T|$.

Construct a sequence $(b_i : i < |T|^+)$ in A:

$\forall i \; \exists \varphi_i(x, b_i) \in tp(a/A)$ which divides / $b_{<i}$

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Moreover, let $\varphi_i(x, b_i)$ divide $/ b_{<i}$ wrt ψ_i.

Since $|\Xi| = |T|$, there is a pair $(\varphi, \psi) \in \Xi$ s.t.

$I = \{ i : (\varphi_i, \psi_i) = (\varphi, \psi) \}$ is infinite.

$\Rightarrow \forall i \in I \quad \varphi(x, b_i)$ div $/ b_{\{j \in I : j < i\}}$ wrt ψ.

Contradicting 2.

3 $\Rightarrow$ 1 by defn.

2 $\Rightarrow$ 4 if $\exists\, \xi \in \Xi^{|T|^+}$, $\xi \in D(p, \Xi)$, then same argument.

Then some pair (φ, ψ) appears infinitely many times in ξ, contradicting 2. (look at a realisation of $\models \mathrm{div}_{\phi, \xi}$).

4 $\Rightarrow$ 2 ~~If 2 is false, then~~ ~~by compactness~~

~~for (φ, ψ)~~

If 2 is false for (φ, ψ), then by compactness,

$\mathrm{div}_{\phi, (\varphi, \psi)^{|T|^+}}$ is consistent $\Rightarrow$ not 4 $\square$.

So from now on, assume T is simple

$\Rightarrow \forall p \quad D(p, \Xi)$ is a set, closed under limits (by def)

$\Rightarrow$ contains maximal element.

[$\xi \leq \zeta$ if ζ is an extension of ξ]

Theorem (T simple) Let $p = tp(a/b)$ and $q = tp(a/bc)$.

TFAE

① $D(p, \Xi) = D(q, \Xi)$.

② $\exists \xi \in D(p, \Xi)$ maximal that is also in $D(q, \Xi)$ (not nec maximal still!).

③ q does not divide over b.

Proof ① $\Rightarrow$ ② ~~maximal~~ maximal elements exist.

② $\Rightarrow$ ③ assume q divides over b.

So $\exists \varphi(x,d) \in q \quad (d \subseteq bc)$ dividing $/b$ wrt some ψ.

$\Rightarrow \xi \in D(q, \Xi) \subseteq D(p \wedge \varphi(x,d), \Xi)$

$\Rightarrow \xi \wedge (\varphi, \psi) \in D(p, \Xi)$ contradicting maximality.

③ $\Rightarrow$ ① (tricky part ☺)

Let $\xi = ((\varphi_i, \psi_i) : i < \alpha)$.

$\cdot$/ $\rangle$

We prove by induction that if $\xi \in D(p, \Xi)$ then $\xi \in D(q, \Xi)$. (converse is clear since $p \leq q$).

To come later.... $\Box$ish.

Cor of ② $\Rightarrow$ ③: Extension is true.

Proof We are given a, b, c.

Let $\xi \in D(\overset{p:}{tp}(a/c), \Xi)$ be maximal.

Since ~~tp(a/c)~~ p is over b, c as a partial type, $p(x) \wedge \operatorname{div}_{\xi, bc}(x)$ is consistent.

Let $a' \models p(x) \wedge \operatorname{div}_{\xi, bc}(x)$.

Then $a' \equiv_c a$ and $a' \underset{c}{\downarrow} b$ since $D(tp(a'/bc), \Xi)$ contains a maximal element of $D(tp(a'/c), \Xi)$. $\Box$

Now since we have extension, we have symmetry, transitivity etc. Still have independence theorem.

Lemma Assume $(a_i : i \leq \omega)$ is c-indiscernible.

Then $a_\omega \underset{a_{<\omega}}{\downarrow} c$.

Proof Let $(c_j : j<\omega)$ be $a_{<\omega}$-indiscernible in $tp(c/a_{<\omega})$. Let $\varphi(x, a_{<n}c) \in tp(a_\omega/a_{<\omega}c)$.

Then $\models \bigwedge_j \varphi(a_n, a_{<n}, c_j)$ (since $\models \varphi(a_n, a_{<n}, c)$)

$\Rightarrow \varphi(x, a_{<n}, c)$ dnd / $a_{<\omega}$ $\square$

Lemma Let $(a_i : i<2\omega)$ be a c-indiscernible sequence. Then $(a_{\omega+i} : i<\omega)$ is a Morley sequence over $c, a_{<\omega}$.

Proof $(a_i : i\leq\omega)$ is indiscernible over $c \cup \{a_j : \omega<j<2\omega\}$

$\Rightarrow a_\omega \underset{a_{<\omega}}{\downarrow} c\, a_{>\omega} \overset{\text{trans}}{\Rightarrow} a_\omega \underset{a_{<\omega}c}{\downarrow} a_{>\omega}$

Notice $0 \cdots \omega, \omega+1, \cdots$
$0 \cdots \omega, \omega+1, \omega+2, \cdots$

$tp(a_{<\omega}, a_{\geq\omega}/c) = tp(a_{<\omega}, a_{\geq\omega+n}/c)$.

$\Rightarrow a_{\omega+n} \underset{c\,a_{<\omega}}{\downarrow} a_{>\omega+n}$

By induction, want to prove: $a_\omega \cdots a_{\omega+n} \underset{a_{<\omega}c}{\downarrow} a_{<\omega+n}$. (*)

For $n=0$ ✓.

3/-

For $n+1$ we have $a_{\omega+n+1} \underset{a_{<\omega}c}{\downarrow} a_{>\omega+n+1}$.

$\circledast^{+\text{trans}} \Rightarrow a_{\omega+\cdots} a_{\omega+n} \underset{c\, a_{<\omega}\, a_{\omega+n+1}}{\downarrow} a_{>\omega+n+1}$

$\overset{\text{trans}}{\Rightarrow}$ $a_{\omega} \dots a_{\omega+n+1} \underset{c\, a_{<\omega}}{\downarrow} a_{>\omega+n+1}$.

So now by symmetry, $\forall m\ a_{\omega+m} \underset{c\, a_{<\omega}}{\downarrow} a_{\omega} \dots a_{\omega+m-1}$. $\square$.

<u>Corollary</u>

Assume that $p(x, b, c)$ does not divide over c.

Let (b_i) be c-indiscernible in $tp(b/c)$.

Then $\bigcup p(x, b_i, c)$ is consistent and dnd $/c$.

[Last few lemmas: Kim "Forking in Simple Theories"].

<u>Proof</u> ~~Since~~ Extend $(b_i : i<\omega)$ to a similar sequence $/c$ $(b_i : i<2\omega)$

By nondividing (e extn), $\exists a$ s.t. $a \models p(x, b_\omega, c)$ and

$a \underset{c}{\downarrow} b_{<\omega}$ [Find $\xi \in D(p(x, b_\omega, c) \Xi_\ast)$ maximal & follow previous proofs ie realise $\mathrm{div}_{c\, b_{\leq\omega}, \xi} \wedge p(x, b_\omega, c)$]

$(b_{\omega+i})$ is $b_{<\omega}c$-indiscernible. $\Rightarrow$ since $a \underset{b_{<\omega}c}{\downarrow} b_\omega$,

~~we~~ may assume that $(b_{\omega+i} : i<\omega)$ is $a, b_{<\omega}, c$-indiscernible

since we can send it to one by an $(b_{<\omega}, c)$-automorphism.

But $(b_{\omega+i} : i<\omega)$ is a Morley sequence over $(b_{<\omega}, c)$.

$\Rightarrow$ by a previous result, since it is also

$a, b_{<\omega}, c$ - indiscernible, we have $a \underset{c b_{<\omega}}{\downarrow} b_\omega, b_{\omega+1}, \ldots$

Now add in $a \underset{c}{\downarrow} b_{<\omega} \Rightarrow a \underset{c}{\downarrow} b_{\geq 2\omega} \Rightarrow a \underset{c}{\downarrow} b_\omega, b_{\omega+1}, \ldots$

We also have $\models p(a, b_{\omega+i}, c) \ \forall i<\omega$.

$\Rightarrow \bigcup p(x, b_{\omega+i}, c)$ does not divide $/c$.

$\Rightarrow \bigcup p(x, b_i, c)$ does not divide $/c$. $\square$

25. Improved Improved Extension

If $p(x, bc)$ is a partial type over bc & dnd$/c$, then it can be extended to a complete type over bc that does not divide $/c$.

Proof By basic extn, $\exists$ Morley sequence (b_i) for b/c.

Since $p(x, bc)$ dnd$/c$ $\exists a' \models \bigwedge p(x, b_i c)$

We may assume (b_i) is $a'c$-indiscernible

$\Rightarrow a' \underset{c}{\downarrow} b_0$.

Then $q(x, b_0, c) := tp(a'/b_0 c)$ dnd$/c$ & $q(x, bc)$ is what we wanted $\square$