$A \subseteq \mathcal{U}$, I set of variable symbols.
↑universal domain.

← type means complete type.

Given $p \subseteq \Delta$, we say p is an I-type over A

if $\exists f: I \to \mathcal{U}$ $\quad p = \{ \varphi(\bar{x}, \bar{a}) : \varphi \in \Delta, \bar{x} \in I^m, \bar{a} \in A^n, \mathcal{U} \models \varphi(f(x), \bar{a}) \}$.

We say $p = tp(f(x) : x \in I / A)$.

$$S_I(A) = \{ I\text{-types over } A \}.$$

$$|I| = |J| \Rightarrow S_I(A) \cong S_J(A)$$

Let $I_0 \subseteq I_1 \subseteq I_2 \subseteq \dots$ & $I = \bigcup_{n<\omega} I_n$

$p_0 \subseteq p_1 \subseteq p_2 \subseteq \dots$ & $p_\omega = \bigcup_{n<\omega} p_n$

Suppose $p_n \in S_{I_n}(A)$. Then $p_\omega \in S_I(A)$.

Proof. Suppose p_ω not realised. Then $\exists \varphi_1, \dots, \varphi_k \in p_\omega$ s.t.

$\{\varphi_1, \dots, \varphi_k\}$ not realised. But $\exists n<\omega$ $\{\varphi_1, \dots, \varphi_k\} \in p_n$ ※.

Suppose $(b_i : i \in I)$ realises p_ω.

Suppose $\mathcal{U} \models \varphi(b_{i_1} \dots b_{i_k}, a_1, \dots, a_m)$ where $i_1, \dots, i_k \in I$.

$\exists n < \omega$ $\quad i_1, \dots, i_k \in I_n$ $\quad : \varphi(x_{i_1}, \dots, x_{i_k}, a_1, \dots, a_m) \in p_n$.

So p_ω a complete type. $\square$

Suppose $\alpha, \delta \in ORD$ (& small) & we have $b_i \in \mathfrak{C}^\delta$ $\forall i < \alpha$.

We say $(b_i : i < \alpha)$ is A-indiscernible if

$\forall i_0 < \cdots < i_{n-1} < \alpha$ $\forall j_0 < \cdots < j_{n-1} < \alpha$

we have $tp(b_{i_0}, \ldots, b_{i_{n-1}} / A) = tp(b_{j_0}, \ldots, b_{j_{n-1}} / A)$.

<u>Set Theory</u> $\beth_0 = \aleph_0$, $\beth_{\alpha+1} = 2^{\beth_\alpha}$, $\beth_\lambda = \bigcup_{\alpha<\lambda} \beth_\alpha$.

more generally, $\beth_0(\kappa) = \kappa$, $\beth_{\alpha+1}(\kappa) = 2^{\beth_\alpha(\kappa)}$, $\beth_\lambda(\kappa) = \bigcup_{\alpha<\lambda} \beth_\alpha(\kappa)$

$cf \lambda = \min \{ \alpha : \exists f \in {}^\alpha\lambda \text{ st. } \sup \operatorname{ran} f = \lambda \}$.

(limit ordinal)

one property: $cf \kappa^+ = \kappa^+$

another: $cf \beth_\lambda(\kappa) = cf \lambda$.

We will use today $cf \beth_{\kappa^+} = cf \kappa^+ = \kappa^+$.

<u>More Notation:</u>
- $[A]^\kappa = \{ B \subseteq A : |B| = \kappa \}$
- Given $n < \omega$ & cardinals λ, μ, κ.

$\boxed{\kappa \rightarrow (\lambda)^n_\mu}$ means for all $f : [\kappa]^n \rightarrow \mu$ $\exists A_f \in [\kappa]^\lambda$ such that f is constant on $[A]^n$.

(Ramsey's Thm: $\omega \rightarrow (\omega)^n_k$ $\forall n, k < \omega$)

<u>Erdős-Rado Thm:</u> $\beth_n(\kappa)^+ \longrightarrow (\kappa^+)^{n+1}_\kappa$. $\square$

<u>Lemma</u>: Let $A \subseteq \mathfrak{U}$, $\lambda \geq |S_\kappa(A)|$.

Set $\mu = \beth_{\lambda^+}$. For each sequence $(a_i : i<\mu)$ of κ-tuples in $\mathfrak{U}$ $\exists (b_i : i<\omega)$ in $\mathfrak{U}^\kappa$ such that $(b_i : i<\omega)$ is A-indiscernible & $\forall n<\omega$ $\exists i_0, \dots, i_{n-1} < \mu$ st. $tp(a_{i_0}, \dots, a_{i_n} / A) = tp(b_0, \dots, b_{n-1} / A)$.

<u>Proof</u> Given $n<\omega$, let x_n be a κ-tuple of variable symbols.

Let $J_n := \bigcup_{i<n} x_i$. and $J := \bigcup_{n<\omega} x_n$.

Suppose $\forall n<\omega$ $\exists p_n \in S_{J_n}(A)$ st. $\forall$ cardinals $\eta<\mu$, we have: (P_n) $p_n \supseteq p_m(x_{i_0}, \dots, x_{i_{m-1}})$ $\forall i_0 < \dots < i_{m-1}$ $\forall m<n$.

a property name

substitute variables $x_0 \dots x_{m-1}$ with $x_{i_0} \dots x_{i_{m-1}}$.

$(Q_{n,\eta})$ $\exists I \in [\mu]^\eta$ $\forall i_0 < \dots < i_{n-1} < \mu$ if $\{i_0, \dots, i_{n-1}\} \subseteq I$ then $(a_{i_0}, \dots, a_{i_{n-1}})$ realises p_n.

another property

By (P_n), $p_n \supseteq p_m(x_0, \dots, x_{m-1}) = p_m$, hence

$$p_0 \subseteq p_1 \subseteq p_2 \subseteq \cdots \quad \therefore \quad p_\omega := \bigcup_{n<\omega} p_n \in S_J(A).$$

Moreover, we have (P_n) for all $n<\omega$,

so if $(b_i : i<\omega)$ realises p_ω, then $(b_i : i<\omega)$ is A-indiscernible: any $i_0 < \dots < i_{m-1} < n$ satisfies (P_n) for p_m, hence $(b_{i_0}, \dots, b_{i_{m-1}})$ realises p_m.

And by $(Q_{n,n})$ for all $n<\omega$,

$\exists\, i_0 < \cdots < i_{n-1} < \mu$ $tp(b_0,\ldots,b_{n-1}/A) = p_n = tp(a_{i_0},\ldots,a_{i_{n-1}}/A)$.

Now ~~proved~~ prove ~~parts~~ proposition by induction.

$n=0$ is vacuous. Assume p_n satisfies (P_n) & $(Q_{n,\eta})$ $\forall \eta<\mu$.

Let $S = \{q \in S_{J_{n+1}}(A) : q \text{ satisfies } (P_{n+1})\}$

If $q \in S$ and q satisfies (Q_{n+1},η) $\forall \eta<\mu$ then we're done.

Suppose $\forall q \in S$ $\exists\, \eta_q < \mu$ s.t. q fails (Q_{n+1},η_q).

Choose such η_q.

Let $\eta := \max(\lambda, \sup\{\eta_q : q \in S\})$ where $\sup \emptyset = 0$.

Then $\operatorname{cf}\mu = \operatorname{cf} \beth_{\lambda^+} = \lambda^+ > \lambda \geq |S_K(A)| = |S_{J_{n+1}}(A)| \geq |S|$.

$\therefore$ $\sup\{\eta_q : q \in S\} < \mu$.

Also $\lambda < \beth_{\lambda^+} = \mu$. $\therefore$ $\eta < \mu$.

$\forall q \in S$ $\eta_q \leq \eta$ $\therefore$ $\forall q \in S$ fails (Q_{n+1},η).

λ^+ is a limit ordinal, hence $\exists\, \theta < \lambda^+$ $\eta < \beth_\theta$.

$\nu = \beth_{\theta+n+1}$ $\therefore$ $\nu < \mu$.

By inductive hypothesis, $\exists\, I \in [\mu]^\nu$ $\forall i_0 < \cdots < i_{n-1} < \mu$ if

$\{i_0,\ldots,i_{n-1}\} \in I$ $(a_{i_0},\ldots,a_{i_{n-1}})$ realises p_n.

By Erdős - Rado, $\beth_n(\eta)^+ \to (\eta^+)^{n+1}_{\eta}$

Now $\nu = \beth_{n+1}(\beth_\theta) \geq \beth_{n+1}(\eta) \geq \beth_n(\eta)^+$.

∴ we have $\nu \to (\eta^+)^{n+1}_{\eta}$

Let $f : [I]^{n+1} \to S_{J_{n+1}}(A)$ be defined by

$f(\{i_0, \dots, i_n\}) := tp(a_{i_0}, \dots, a_{i_n}/A)$ where $i_0 < \dots < i_n$.

$\exists\, I' \in [I]^{\eta^+}$ s.t. f is constant on I'.

| Reminder: $|S_{J_{n+1}}(A)| \leq \lambda \leq \eta$ |

Choose $j_0, \dots, j_n \in I'$ s.t. $j_0 < \dots < j_n$.

Set $q = tp(j_0, \dots, j_n/A) \in S_{J_{n+1}}(A)$.

Then I' witnesses that q satisfies (Q_{n+1}, η^+).

Also, $\{j_0, \dots, j_n\} \in I$; hence $\forall\, i_0 < \dots < i_{n-1} < n+1$ we have $(a_{j_0}, \dots, a_{j_{i_{n-1}}})$ realises p_n.

On the otherhand $\forall\, m < n\ \forall\, i_0 < \dots < i_{m-1} < n$, then we have $(a_{j_{i_0}}, \dots, a_{j_{i_{m-1}}})$ realises p_m (by (P_n)).

∴ q satisfies (P_{n+1}). ∴ $q \in S$.

∴ q fails (Q_{n+1}, η_q) But $\eta_q < \eta^+$ ※.

Convention: All indiscernible sequences are infinite.

Definition: We say that two A-indiscernible sequences $(a_i : i<\alpha)$, $(b_j : j<\beta)$ are similar (over A) if $\forall n<\omega$, $tp(a_0,\dots,a_{n-1}) = tp(b_0,\dots,b_{n-1})$.

$\left(\Leftrightarrow tp(\underset{\substack{\| \\ \{a_0,\dots,a_i\}_{i<\omega}}}{a_{<\omega}}/A) = tp(b_{<\omega}/A) \Leftrightarrow \forall i_0<\dots<i_{n-1}<\alpha,\ j_0<\dots<j_{n-1}<\beta\ tp(a_{i_0},\dots,a_{i_{n-1}}/A) = tp(b_{j_0},\dots,b_{j_{n-1}}/A)\right)$

(So the only difference btwn the two is the length)

Easy Fact: Assume $(a_i : i<\omega)$ is A-indiscernible.

Then $\forall \lambda \geq \omega$ there is an A-indiscernible sequence $(b_i : i<\lambda)$ similar to (a_i);

moreover, $tp((b_i : i<\lambda)/A)$ is uniquely determined by $tp((a_i)/A)$ and λ

Proof Define $\forall n$: $p_n = tp(a_0,\dots,a_{n-1}/A)$.

$$q(x_{<\lambda}) = \bigcup_{n<\omega} \bigcup_{i_0,\dots,i_{n-1}<\lambda} p_n(x_{i_0},\dots,x_{i_{n-1}})$$

Then q is consistent by compactness.

$[\, p_n(x_{i_0}\dots\widehat{x_{i_j}}\, x_{i_n}) \subseteq p_{n+1}(x_0,\dots,x_{i_n}) \,]$

and is a complete type. (use fact we have directed set of indices).

clearly any realisation of q will do & only of q $\square$

<u>Corollary</u> Let $A \subseteq B$. $(a_i : i<\omega)$ an A-indiscernible sequence.
Then there exists a B-indiscernible sequence $(b_i : i<\omega)$
which is similar to $(a_i)/A$.

<u>Proof</u> Extension/extraction technique.

First, by the Fact, there is a similar sequence ~~$(b_i : i<\lambda)$~~ $(b_i : i<\mu)$ $\forall \mu$.
So choose μ big enough for the "Erdös - Rado Lemma."
Then there is a sequence $(c_i : i<\omega)$ B-indiscernible s.t.
$\forall n \;\; \exists\, i_0 < \cdots < i_{n-1} < \mu$ s.t. $tp(c_0, \ldots, c_{n-1}/B) = tp(b_{i_0}, \ldots, b_{i_{n-1}}/B)$
Since $B \supseteq A$, $tp(c_0, \ldots, c_{n-1}/A) = tp(b_{i_0}, \ldots, b_{i_{n-1}}/A)$
$= tp(a_0, \ldots, a_{n-1}/A)$. $\square$