9 Lecture 9 (Notes: K. Venkatram)

Last time, we talked about the geometry of a connected lie group G. Specifically, for any a in the corresponding Lie algebra $\mathfrak{g}$, one can define $a^L|_g = L_{g*}a$ and choose $\theta^L \in \Omega^1(G, \mathfrak{g})$ s.t. $\theta^L(a^L) = a$. For instance, for GL_n, with coordinates $g = [g_{ij}]$, one has $\theta^L = g^{-1}dg$, and similarly $\theta^R = dgg^{-1}$.This implies that $dg \wedge \theta^L + gd\theta^L = 0 \implies d\theta^L + \theta^L \wedge \theta^L = 0 \implies d\theta^L + \frac{1}{2}[\theta^L, \theta^L] = 0$, the latter of which is the Maurer-Cartan equation.

Problem.

1. Extend this proof so that it works in the general case.
2. Show $j^*\theta^R = -\theta^L$.
3. Show $d\theta^R - \frac{1}{2}[\theta^R, \theta^R] = 0$.
4. Show $\theta^R(a^L)|_g = \mathrm{Ad}\ {}_g a \forall a \in \mathfrak{g}, g \in G$.

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9.1 Bilinar forms on groups

Let G be a connected real Lie group, B a symmetric nondegenerate bilinear form on $\mathfrak{g}$. This extends to a left-invariant metric on G, and B is invariant under right translation
$\Leftrightarrow B([X,Y],Z) + B(Y,[X,Z]) = 0 \forall X,Y,Z$. If this is true, we obtain a bi-invariant (pseudo-Riemannian) metric on G.

Remark. Geodesics through e are one-parameter subgroups $\Leftrightarrow B$ is bi-invariant. See Helgason for Riemannian geometry of Lie groups and homogeneous spaces.

Example. Let B be the Killing form on a semisimple Lie group, i.e. $B(a,b) = \mathrm{Tr}_g(\mathrm{ad}_a \mathrm{ad}_b)$ for $\mathfrak{s}|_m, \mathfrak{s} \circ m, \mathfrak{s}p_m$ a constant multiple of $\mathrm{Tr}(X,Y)$. Now, we can form the Cartan 3-form

$$H = \frac{1}{12} B(\theta^L, [\theta^L, \theta^L]) = \frac{1}{12} B(\theta^R, [\theta^R, \theta^R]) \tag{7}$$

This H is bi-invariant, and thus closed. When G is simple, compact, and simply connected, the Killing form gives $\lambda[H]$ as a generator for $H^3(G,\mathbb{Z}) = \mathbb{Z}$. (See Brylinski.) For instance, given $\mathfrak{g} = \mathfrak{s}|_n, \theta^L = g^{-1}dg$, one has $H = \mathrm{Tr}(\theta^L \wedge \theta^L \wedge \theta^L)$ i.e. $H = \mathrm{Tr}(g^{-1}dg)^3$.

9.1.1 Key calculation

Let $m, p_1, p_2 : G \times G \to G$ be the multiplication and projection maps respectively. Then

$$\begin{aligned} m^*H &= \mathrm{Tr}((gh)^{-1}d(gh))^3 = \mathrm{Tr}(h^{-1}g^{-1}(gdh + dgh))^3 \\ &= \mathrm{Tr}(h^{-1}gh)^3 + \mathrm{Tr}(g^{-1}dg)^3 + \mathrm{Tr}((dhh^{-1})^2 g^{-1}dg) + \mathrm{Tr}(dhh^{-1}(g^{-1}dg)^2) \end{aligned} \tag{8}$$

Now, define $\theta = dhh^{-1}, \Omega = g^{-1}dg$, so $d\theta = \theta \wedge \theta$ and $d\Omega = -\Omega \wedge \Omega$. Then

$$\begin{aligned} d\mathrm{Tr}(dhh^{-1}g^{-1}dg) = d\mathrm{Tr}(\theta \wedge \Omega) &= \mathrm{Tr}(d\theta \wedge \Omega - \theta \wedge d\Omega) \\ &= \mathrm{Tr}(\theta \wedge \theta \wedge \Omega + \theta \wedge \Omega \wedge \Omega) \end{aligned} \tag{9}$$

So, $m^*H - p_1^*H - p_2^*H = d\tau$, where $\tau = \mathrm{Tr}(dhh^{-1}g^{-1}dg) = B(p_1^*\theta^L, p_2^*\theta^R) \in \Omega^2(G \times G)$.
Now, recall that given a metric $g : V \to V^*$, we have a decomposition $V \oplus V^* = C_+ \oplus C_-$ for $C_\pm = \Gamma_\pm$. Moreover, any Dirac structure $L \subset V \oplus V^*$ can be written as the graph of $A \in O(V, \mathfrak{g})$ thought of as $A : C_+ \to C_-$. NOw, for $X \in V$, let $X^\pm = X \pm gX \in C_\pm$. Then $L^A_\pm = \{X^+ \pm (AX)^- | X \in V\}$ are the Dirac structures. Note that

$$\langle X^+ \pm (AX)^-, X^+ \pm (AX)^- \rangle = g(X,X) - g(AX,AX) = 0 \tag{10}$$

Let B be a bi-invariant metric on G. Then the map $A_x = L_{x^{-1}*}R_{x*} : T_xG \to T_xG, a^L \mapsto a^R$ is orthogonal for B and $\mathrm{ad}(G)$-invariant, since

$$\begin{tikzcd} T_xG \arrow[r, "A_x"] \arrow[d, "\mathrm{ad}_{g*}"'] & T_xG \arrow[d, "\mathrm{ad}_{g*}"] \\ T_{gxg^{-1}}G \arrow[r, "A_{gxg^{-1}}"] & T_{gxg^{-1}}G \end{tikzcd} \tag{11}$$

where $\mathrm{ad}_{g*} = L_{g*}R_{g^{-1}*}$. Thus, we find that

$$\mathrm{ad}_{g*}A_x\mathrm{ad}_{g*}^{-1} = L_gR_{g^{-1}}R_xL_{x^{-1}}R_gL_{g^{-1}} = L_{g^{-1}x^{-1}g}R_{gxg^{-1}} = A_{gxg^{-1}} \tag{12}$$

Overall, $L_\pm(A)$ are $\mathrm{ad}(G)$-invariant almost Dirac structures in $(T \oplus T^*)(G)$. T_xG is spanned by the a^L, so L_+ is spanned by $(a^L)^+ + (a^L)^- = a^L + B(a^L) + a^r - B(a^R)$ and $L_+ = \langle a^L + a^R + B(a^L - a^R) \rangle$. Recall that $\theta^L(a^L) = a$ so $\langle a^L + a^R + B(a^L - a^R) \rangle = \langle a^L + a^R + B(\theta^L - \theta^R, a) \rangle$. Similarly,
$L_- = \langle a^L - a^R + B(\theta^L + \theta^R, a) \rangle$.

Remark. Since $a^L - a^R$ generates the adjoint action, $[a^L - a^R, b^L - b^R] = [a,b]^L - [a,b]^R$. But $[a^L + a^R, b^L + b^R] = [a,b]^L + [a,b]^R$ is not integrable. $L_-(A)$ is integrable, however, w.r.t. the Courant bracket twisted by $H = B(\theta^L, [\theta^L, \theta^L])$.