SYMPLECTIC GEOMETRY, LECTURE 17

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The Hodge decomposition stated last time places strong constraints on H^* of Kähler manifolds, e.g. dim H^k is even for k odd because $\mathbb{C}$ conjugation gives isomorphisms $\overline{\mathcal{H}^{p,q}} \cong \mathcal{H}^{q,p}$ (note that this is false for symplectic manifolds in general). The Hodge star $*$ gives isomorphisms $\mathcal{H}^{p,q} \xrightarrow{\sim} \mathcal{H}^{n-q,n-p}$ and the *Hodge diamond structure* on the the ranks of the Dolbeault cohomology groups, i.e.

$$\begin{array}{cccc} h^{n,n} & \cdots & \cdots & h^{0,n} \\ \vdots & \ddots & \ddots & \vdots \\ \vdots & \ddots & h^{1,1} & h^{0,1} \\ h^{n,0} & \cdots & h^{1,0} & h^{0,0} \end{array} \tag{1}$$

is symmetric across the two diagonal axes. Moreover, note that $[\omega^{\wedge p}] \in \mathcal{H}^{p,p}$ is nonzero, since $[\omega^{\wedge n}]$ is the volume class.

We have even stronger constraints, namely the "hard Lefschetz theorem".

Theorem 1. $L^{n-k} = (\cdot \wedge \omega^{n-k}) : H^k(X, \mathbb{R}) \to H^{2n-k}(X, \mathbb{R})$ *is an isomorphism.*

This is false for many symplectic manifolds. Moreover, combining this with Poincaré duality gives that, for $k \leq n$, $H^k \times H^k \to \mathbb{R}$, $\alpha, \beta \mapsto \int \alpha \cup \beta \cup \omega^{n-k}$ is a nondegenerate bilinear pairing (skew-symmetric if k is odd). We also have the *Kodaira embedding theorem*:

Theorem 2. *For (X, ω) a compact Kähler manifold, $[\omega] \in H^2(X, \mathbb{Z})$, $\exists$ a projective embedding $X \to \mathbb{CP}^N$ realizing X as a projective algebraic variety.*

We will see a symplectic geometry proof due to Donaldson.

1. Holomorphic vector bundles

Let (M, J) be a complex manifold, $E \to M$ a complex vector bundle. Then we can cover M by U_α s.t. the restrictions $U_\alpha \times \mathbb{C}^n \cong E|_{U_\alpha} \to U_\alpha$ are trivial.

Definition 1. *E is a holomorphic vector bundle if the transition functions $\phi_{\alpha,\beta} : U_\alpha \cap U_\beta \to \mathrm{GL}(r, \mathbb{C})$ are holomorphic.*

Note that this only makes sense on a complex manifold. Now, $\exists$ a natural $\overline{\partial}$ operator on sections given in a local trivialization by $\overline{\partial}$ (given a section s which looks like ξ_α in the local trivialization α, on an intersection we have that $\overline{\partial}\xi_\alpha = \phi_{\alpha,\beta}\overline{\partial}\xi_\beta$ since $\overline{\partial}\phi_{\alpha,\beta} = 0$). This extends to $\overline{\partial} : \Omega^{p,q}(E) \to \Omega^{p,q+1}(E)$ similarly.

Definition 2. $H^q_{\overline{\partial}}(E) = \frac{\mathrm{Ker}\ (\overline{\partial}:\Omega^{0,q}(E)\to\Omega^{0,q+1}(E))}{\mathrm{Im}(\overline{\partial}:\Omega^{0,q-1}(E)\to\Omega^{0,q}(E))}$. *In particular, $H^0(E)$ is the space of holomorphic sections.*

Specifying the holomorphic structure on a complex vector bundle E is equivalent to specifying a $\overline{\partial}$ operator with $\overline{\partial}^2 = 0$. The $\overline{\partial}$ operator is half of a connection: in fact, ∇ a connection on E decomposes into $\nabla = \nabla^{1,0} + \nabla^{0,1}$.

Proposition 1. *For $(E, \overline{\partial}, |\cdot|)$ a holomorphic bundle with a Hermitian metric, $\exists!$ Hermitian connection s.t. $\nabla^{0,1} = \overline{\partial}$.*

Proof. We work in local coordinates on M, and local trivializations of E by orthonormal sections σ_j (but not necessarily holomorphic trivializations; $\overline{\partial}\sigma_j$ may be nonzero). $\nabla = d + A$ for $A = (a_{ij})$ a matrix-valued 1-form ($a_{ij} = \langle \nabla\sigma_j, \sigma_i\rangle$). ∇ is Hermitian iff $a_{ij} = -\overline{a_{ij}}$, i.e. A is antihermitian, and ∇ is holomorphic, i.e. $\nabla^{0,1}s = \overline{\partial}s$ iff $A^{0,1}$ is given by $a^{0,1}_{ij} = \langle\overline{\partial}\sigma_j, \sigma_i\rangle$. Then $A^* = -A \Leftrightarrow A^{1,0} = -(A^{0,1})^*$, i.e. $a^{1,0}_{ij} = -\overline{a^{0,1}_{ji}}$. $\square$

Equivalently, in a holomorphic trivialization, when $\overline{\partial}$ is the usual $\overline{\partial}$ operator, $\langle\cdot,\cdot\rangle$ given by $h = C^\infty$ function with values in positive definite Hermitian matrices, $\nabla = d + A$ again and ∇ is Hermitian $\Leftrightarrow d\langle s, s'\rangle = \langle \nabla s, s'\rangle + \langle s, \nabla s'\rangle \Leftrightarrow d(s^* h s') = (ds^* + s^* A^*) h s' + s^* h(ds' + As') \Leftrightarrow dh = A^* h + hA$. On the other hand, now $\nabla^{0,1} = \overline{\partial} \Leftrightarrow A^{0,1} = 0$. Hence $dh = A^* h + hA \Leftrightarrow A = h^{-1}\partial h$ (and $A^* = \overline{\partial} h \cdot h^{-1}$).

Proposition 2. *In a holomorphic frame, the connection* 1*-form* A *is of type* $(1,0)$*, and* $\partial A = -A \wedge A$*,* $R^\nabla = \overline{\partial} A$ *is of type* $(1,1)$*, and* $\overline{\partial} R = 0$ *and* $\partial R = [R, A]$*.*

In fact, we have

Theorem 3. $(E, \nabla^{0,1} = \overline{\partial}^\nabla)$ *is holomorphic* $\Leftrightarrow (\overline{\partial}^\nabla)^2 = 0 \Leftrightarrow R^{0,2} = 0$.

Proof. First, $A = h^{-1}\partial h$ has type $(1,0)$ by the above, and

$$\partial A = \partial(h^{-1}) \wedge \partial h = (-h^{-1}(\partial h) h^{-1}) \wedge \partial h = -(h^{-1}\partial h) \wedge (h^{-1}\partial h) = -A \wedge A \tag{2}$$

by the formula for derivatives of inverses in a noncommutative setting. Second, $R^\nabla = dA + A \wedge A = dA - \partial A = \overline{\partial} A$, hence it has type $(1,1)$. Finally, $\overline{\partial} R = \overline{\partial}\overline{\partial} A = 0, \partial R = \partial\overline{\partial} A = -\overline{\partial}\partial A = \overline{\partial} A \wedge A - A \wedge \overline{\partial} A = [R, A]$. □