SYMPLECTIC GEOMETRY, LECTURE 8

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1. Almost-complex Structures

Recall *compatible triples* (ω, g, J), wherein two of the three determine the third $(g(u,v) = \omega(u, Jv), \omega(u,v) = g(Ju, v), J(u) = \tilde{g}^{-1}(\tilde{\omega}(u))$ where $\tilde{g}, \tilde{\omega}$ are the induced isomorphisms $TM \to T^*M)$.

Proposition 1. *For (M, ω) a symplectic manifold with Riemannian metric g, $\exists$ a canonical almost complex structure J compatible with ω.*

Idea. Do polar decomposition on every tangent space. □

Corollary 1. *Any symplectic manifold has compatible almost-complex structures, and the space of such structures is path connected.*

Proof. For the first part, using a partition of unity gives a Riemannian metric, so the rest follows from the proposition. For the second part, given J_0, J_1, let $g_i = \omega(\cdot, J_i \cdot)$ for $i = 0, 1$ and set $g_t = (1-t)g_0 + tg_1$. Each of these (for $t \in [0,1]$) is a metric, and gives an ω-compatible $\tilde{J}_t$ by polar decomposition, with $\tilde{J}_0 = J_0$ and $\tilde{J}_1 = J_1$. □

The mechanism of the proof also gives

Proposition 2. *The set $\mathcal{J}(T_xM, \omega_x)$ of ω_x-compatible complex structures on T_xM is contractible, i.e. $\exists h_t : \mathcal{J}(T_xM, \omega_x) \to \mathcal{J}(T_xM, \omega_x)$ for $t \in [0,1], h_0 = \mathrm{id}, h_1 = \mathcal{J} \to J_0, h_t(J_0) = J_0 \forall t$.*

Corollary 2. *The space of compatible almost-complex structures on (M, ω) is contractible. It is the space of sections of a bundle whose fibers are contractible by the previous proposition.*

More generally, let $E \to M$ be a vector bundle.

Definition 1. *A* metric *on E is a family of positive-definite scalar products $\langle \cdot, \cdot \rangle_x : E_x \times E_x \to \mathbb{R}$. E is* symplectic *(resp.* complex*) if there is a family of nondegenerate skew-symmetric forms $\omega_x : E_x \times E_x \to \mathbb{R}$ (resp. complex structures $J_x : E_x \to E_x, J_x^2 = -1$).*

Then metrics always exist, and every sympletic vector bundle is a complex vector bundle and vice versa.

Proposition 3. *For (M, J) an almost-complex manifold, ω_0, ω_1 two symplectic forms compatible with J, $\omega_t = (1-t)\omega_0 + t\omega_1$ is symplectic and J-compatible $\forall t \in [0,1]$ (i.e. the space of J-compatible ω is convex).*

Note that

- The space of such ω might be empty, as there are almost complex manifolds (like S^6) which have no symplectic structures.
- Not every manifold has an almost-complex structure (e.g. S^4, by the Ehresman-Hopf theorem).

  Problem. $\exists$ an almost-complex structure $\Leftrightarrow \exists$ a nondegenerate 2-form.
- The proposition works if we put *tame* instead of compatible, i.e. require $\omega(u, Ju) > 0 \ \forall u \neq 0$ but not symmetry.

Proof. ω_t is closed and $\omega_t(u, Ju) = (1-t)\omega_0(u, Ju) + t\omega_1(u, Ju) > 0 \ \forall u \neq 0$, so ω_t is nondegenerate and thus symplectic. Moreover, $g_t(u,v) = \omega_t(u, Jv) = (1-t)g_0(u,v) + tg_1(u,v)$ is a metric. □

Definition 2. *$X \subset (M, J)$ is an almost-complex submanifold if $J(TX) = TX$, i.e. $\forall x \in X, v \in T_xX, Jv \in T_xX$.*

Proposition 4. *If X is an almost-complex submanifold in compatible (M, ω, J), then X is symplectic (i.e. $\omega|_X$ is nondegenerate).*

Proof. $\forall u \in T_xX, u \neq 0, Ju \in T_xX$ and $\omega(u, Ju) > 0$, so $\forall u \in T_xX \setminus \{0\}, \omega(u, \cdot)|_{T_xX} \in T_x^*X$ is nonzero, giving us an isomorphism $TX \to T^*X$ as desired. □

Let $(\mathbb{R}^{2n}, \Omega_0, J_0, g_0)$ be the standard symplectic structure, complex structure, and metric on $\mathbb{R}^{2n}$.

- $\mathrm{Sp}(2n, \mathbb{R})$ is the group of linear symplectomorphisms of $(\mathbb{R}^{2n}, \Omega_0)$, i.e. $\{A \in GL(2n, \mathbb{R}) | \Omega_0(Au, Av) = \Omega(u, v)\ \forall u, v\}$.
- $\mathrm{GL}(n, \mathbb{C})$ is the group of $\mathbb{C}$-linear automorphisms of $(\mathbb{R}^{2n}, J_0)$, i.e. $\{A | AJ_0 = J_0 A\}$.
- $O(2n)$ is the group of isometries of $(\mathbb{R}^{2n}, g_0)$, i.e. $\{A | A^t A = 1\}$.
- $U(n) = \mathrm{GL}(n, \mathbb{C}) \cap O(2n)$.

Proposition 5. $\mathrm{Sp}(2n) \cap O(2n) = \mathrm{Sp}(2n) \cap \mathrm{GL}(n, \mathbb{C}) = O(2n) \cap \mathrm{GL}(n, \mathbb{C}) = U(n)$.

Proof. The intersection of any two of these sets is the set of automorphisms preserving two of the three in a compatible triple, and thus must preserve all of them. □

- For (V, Ω, J) a symplectic vector space with compatible almost-complex structure, $\exists$ an isomorphism $(V, \Omega, J) \xrightarrow{\sim} (\mathbb{R}^{2n}, \Omega_0, J_0)$.
- The space $\Omega(V)$ of all symplectic structures on V is $\cong \mathrm{GL}(V)/\mathrm{Sp}(V, \Omega_0) \cong \mathrm{GL}(2n, \mathbb{R})/\mathrm{Sp}(2n)$, as $GL(V)$ acts transitively on $\Omega(V)$ by $\phi \mapsto \phi^*\Omega_0$ with stabilizer $\mathrm{Sp}(V, \Omega)$.
- The space $\mathcal{J}(V)$ of almost-complex structures on V is $\cong \mathrm{GL}(V)/\mathrm{GL}(V, J) \cong \mathrm{GL}(2n, \mathbb{R})/\mathrm{GL}(n, \mathbb{C})$.
- The space $\mathcal{J}(V, \Omega)$ of Ω-compatible J's on V is $\cong \mathrm{Sp}(V, \Omega)/\mathrm{Sp}(V, \Omega) \cap GL(V, J) \cong \mathrm{Sp}(2n, \mathbb{R})/U(n)$.
- The constractibility of $\mathcal{J}(V, \Omega)$ is now the fact that $\mathrm{Sp}(2n, \mathbb{R})$ retracts onto its subgroup $U(n)$.

2. Vector Bundles and Connections

For $E \to M$ a real or complex vector bundle, we have an exact sequence

$$0 \to E_x \to T_pE \overset{d\pi}{\to} T_xM \to 0 \tag{1}$$

for each $p \in E, x = \pi(p)$. Here, $E_x \subset T_pE$ gives the set of vertical directions: we would like a splitting $T_pE = E_x \oplus (T_pE)^{horiz}$, i.e. a way to transport from one fiber to another. The data required to do this is a connection.

Definition 3. *A* connection ∇ *on E is an $\mathbb{R}$ or $\mathbb{C}$-linear mapping $C^\infty(M, E) \to C^\infty(M, T^*M \otimes E) = \Omega^1(M, E)$ s.t. $\nabla(f\sigma) = df \cdot \sigma + f\nabla\sigma$. For $v \in T_xM$, we let ∇_v denote the mapping $\sigma \mapsto \nabla\sigma(v)$.*

Choose a local trivialization of E, i.e. a frame of sections e_i s.t. $\mathbb{R}^r$ (or $\mathbb{C}^r$)$\times U \cong E|_U, (\xi_1, \ldots, \xi_r) \mapsto \sum \xi_i e_i$. Then $\nabla\sigma = \nabla(\sum \xi_i e_i) = \sum (d\xi_i)e_i + \xi_i \nabla e_i$, i.e. locally $\nabla = d + A$, where $A = (a_{ij}) \in \Omega^1(M, \mathrm{End}(E))$ is a matrix-valued 1-form (the *connection 1-form*) with a_{ij} the component of ∇e_j along e_i. Globally, given $\nabla, \nabla', \nabla(fs) - \nabla'(fs) = f(\nabla s - \nabla' s)$, so $\nabla - \nabla'$ is $C^\infty(M, E)$-linear and the space of connections is an affine space modeled on $\Omega^1(M, \mathrm{End}(E))$.

2.1. Horizontal Distribution.

Let $\sigma : M \to E$ be a section, $d_x\sigma : T_xM \to T_{\sigma(x)}E$ the induced map. Then $\nabla\sigma(x) \in T_x^*M \otimes E_x$ depends only on $d\sigma(x)$. Thus, we can also think of ∇ as a projection $\pi^\nabla : T_{\sigma(x)}E \to E_x$, with $\nabla_v\sigma = \pi^\nabla(d\sigma(v))$. Then $\mathcal{H}^\nabla = \mathrm{Ker}\ \pi^\nabla$ is the horizontal subspace at $p(x)$.

Definition 4. *For $\langle \cdot, \cdot \rangle$ a Euclidean or Hermitian metric on E, ∇ is* compatible with the metric *if $d\langle \sigma, \sigma' \rangle = \langle \nabla\sigma, \sigma' \rangle + \langle \sigma, \nabla\sigma' \rangle$.*

As above, locally one can find an orthonormal frame of sections (e_i), $\langle e_i, e_j \rangle = \delta_{i,j}$. Writing $\nabla = d + A$ in this trivialization, the compatibility becomes

$$\langle \nabla\xi, \eta \rangle + \langle \xi, \nabla\eta \rangle = \langle d\xi, \eta \rangle + \langle A\xi, \eta \rangle + \langle \xi, d\eta \rangle + \langle \xi, A\eta \rangle \tag{2}$$

Since $d\langle \xi, \eta \rangle = \langle d\xi, \eta \rangle + \langle \xi, d\eta \rangle$, this means that the connection 1-form A must be skew-symmetric (or anti-Hermitian).

Also note that ∇ on E induces a ∇^* on E^* by $d(\phi(\sigma)) = \langle \nabla^*\phi, \sigma \rangle + \langle \phi, \nabla\sigma \rangle$, and similarly for $E \otimes F$, etc.