18.966 – Homework 3 – due Thursday April 19, 2007.

1. Let (M, ω) be a symplectic manifold, J a compatible almost-complex structure, and g the corresponding Riemannian metric. Show that two-dimensional almost-complex submanifolds of M are absolutely volume minimizing in their homology class, i.e.: let C, C' be two-dimensional compact closed oriented submanifolds of M, representing the same homology class $[C] = [C'] \in H_2(M, \mathbb{Z})$. Assume that $J(TC) = TC$ (and the orientation of C agrees with that induced by J). Then $\mathrm{vol}_g(C) \leq \mathrm{vol}_g(C')$.

Hint: compare $\omega_{|C'}$ and the area form induced by g.

2. We will admit the fact that the cohomology ring of $\mathbb{CP}^n$ (the set of complex lines through 0 in $\mathbb{C}^{n+1}$) is $H^*(\mathbb{CP}^n, \mathbb{Z}) = \mathbb{Z}[h]/h^{n+1}$, where $h \in H^2(\mathbb{CP}^n, \mathbb{Z})$ is Poincaré dual to the homology class represented by a linear $\mathbb{CP}^{n-1} \subset \mathbb{CP}^n$.

The tautological line bundle $L \to \mathbb{CP}^n$ is the subbundle of the trivial bundle $\mathbb{C}^{n+1} \times \mathbb{CP}^n$ whose fiber at a point of $\mathbb{CP}^n$ is the corresponding line in $\mathbb{C}^{n+1}$. The homogeneous coordinates on $\mathbb{CP}^n$ are actually sections of the dual bundle L^*. (Convince yourself of this).

a) Show that $c_1(L) = -h$, and show that the direct sum of $T\mathbb{CP}^n$ with the trivial line bundle $\underline{\mathbb{C}}$ is isomorphic to the direct sum of $n+1$ copies of L^*. From this, deduce the Chern classes of the tangent bundle $T\mathbb{CP}^n$.

Hint: show that there is a surjective bundle homomorphism $\mathrm{Hom}(L, \underline{\mathbb{C}}^{n+1}) \to T\mathbb{CP}^n$. What is the kernel?

b) Let $X \subset \mathbb{CP}^n$ be a smooth complex hypersurface of degree d, i.e. the submanifold defined by the equation $P(z_0, \dots, z_n) = 0$ where P is a homogeneous polynomial of degree d (transverse to the zero section, i.e. with nonvanishing differential along its zero set). Show that $T\mathbb{CP}^n{}_{|X} = TX \oplus (L^*)^{\otimes d}_{|X}$, and deduce the Chern classes of TX.

3. Let M be a compact oriented 4-manifold, equipped with a Riemannian metric g. A 2-form is said to be *selfdual* if $*\alpha = \alpha$, *antiselfdual* if $*\alpha = -\alpha$. The bundles of selfdual (resp. antiselfdual) 2-forms are denoted by $\Lambda^2_+ T^*M$ and $\Lambda^2_- T^*M$ respectively.

a) Show that the Hodge $*$ operator induces a decomposition of the space of harmonic forms $\mathcal{H}^2 = \mathcal{H}^2_+ \oplus \mathcal{H}^2_-$ into selfdual and antiselfdual harmonic forms. Show that, with respect to the intersection pairing $(\alpha, \beta) \mapsto \int_M \alpha \wedge \beta$, these summands are definite positive (resp. definite negative) and orthogonal to each other.

b) Assume that (M, ω) is a compact Kähler manifold of real dimension 4. Show that $\Lambda^2_+ T^*M \otimes \mathbb{C} = \Lambda^{2,0} \oplus \Lambda^{0,2} \oplus \mathbb{C}\omega$, where the summands are orthogonal to each other, and $\Lambda^2_- T^*M \otimes \mathbb{C} = \omega^\perp \subset \Lambda^{1,1}$. Deduce that the space of real harmonic (1,1)-forms is $\mathcal{H}^{1,1}_{\mathbb{R}} = \mathcal{H}^2_- \oplus \mathbb{R}\omega$.

(Since algebraic curves in a complex projective surface are Poincaré dual to classes in $NS := H^{1,1}(M) \cap H^2(M, \mathbb{Z})$, this implies the *Hodge index theorem*, which asserts that the intersection pairing on algebraic cycles in a complex projective surface has signature $(1, \dim NS - 1)$).