**Lecture 15.**

# 16 A brief introduction to linear analysis

In a number of place we've talked about the so called infinite dimensional context. In this section we'll introduce briefly the basic notions necessary to discuss this story rigorously. The main application we have in mind is to the

## 16.1 Basic definitions

**Definition 16.1.** A normed vector space is a vector space $X$ (over the real or complex numbers) with a function $\|\cdot\| : X \to \mathbb{R}_+$ satisfying the usual properties of a norm. A Banach space is a complete normed vector space that is all sequences which are Cauchy with respect to the converge.

*Examples.* $C^0(X)$, the space of continuous functions on a compact metric space is a Banach space with its natural norm. Completeness is the statement that a uniform limit of continuous functions is continuous.

$C^k(X)$, the space of k-times continuously differentiable functions on a compact manifold when given the norm

$$\|f\|_{C^k} = \sup_{x\in X,\ I \text{with} \ell(I)\le k} \|\frac{\partial^I f}{\partial x^I}\|.$$

where $I = (i_1, i_2, \dots, i_n)$ is a multi-index and $\ell(I) = \sum_{j=1}^n i_j$. Completeness follows form the same theorem applied to the derivatives of $f$.

$L^p$-spaces.

Spaces of Hölder continuous functions.

Next we wish to consider functions on normed vector spaces. It turns out that continuity of maps on a normed vector space is equivalent to boundedness. More precisely we have:

**Definition 16.2.** A linear map $T : X \to Y$ is called bounded if there is a constant $C \geq 0$ so that for all $x \in X$ we have

$$\|Tx\|_Y \leq C\|x\|_X.$$

Furthermore the smallest such constant $C$ is called the operator norm of $T$ and is denoted $\|T\|$.

**Exercise:** $T : X \to Y$ is continuous if and only $T$ is bounded.

A basic fact of life is that every normed vector space sits in canonical fashion in a Banach space.

**Theorem 16.3.** *To each normed vector space $X$ there corresponds a unique Banach space $\overline{X}$ called the completion of $X$ and a unique injective map continuous linear map $X \to \overline{X}$ satisfying the following universal property. If $T : X \to Y$ is a continuous linear map then there is a unique continuous linear map $\overline{T} : \overline{X} \to \overline{Y}$ so that the operator norm of $T$ and $\overline{T}$ agree.*

For proof see for example Royden's text. In practice the significance of this theorem is that we will consider various norms on $C_0^\infty(\mathbb{R}^n)$ and take the completions with respect to these norms. To check if maps between these completions are continuous it suffices to check that the map is bounded on $C_0^\infty$ with respect to the norms in question.

**Definition 16.4.** Let $\mathcal{B}(X, Y)$ denote the space of bounded linear operators from $X$ to $Y$.

$\mathcal{B}(X, Y)$ is Banach space in its own right. In fact it is a Banach algebra (i.e. a Banach space with the structure of an algebra so that for $x, y \in X$ we have $\|xy\| \leq \|x\|\|y\|$.

### 16.1.1 The three pillar's of linear analysis

You can look in any book on Functional analysis for this material. Its also in Abraham-Marsden and Ratiu.

**Theorem 16.5.** *The Hahn-Banach theorem Let $X$ be a linear space over $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$ and $p : X \to \mathbb{R}$ be a map satisfying*

1. *For all $x, y \in X$ $p(x + y) \leq p(x) + p(y)$*
2. *For all $\lambda \in \mathbb{F}$ and all $x \in X$ we have $p(\lambda x) = |\lambda| p(x)$.*

*Let $Z \subset X$ be a linear subspace and $\rho : Z \to \mathbb{F}$ be a linear functional. If for all $z \in Z$ we have $|\rho(z)| \leq p(z)$ then there is a linear functional $\tilde{\rho} : X \to \mathbb{F}$ which extends $\rho$ and satisfies $|\tilde{\rho}(x)| \leq p(x)$ for all $x \in X$.*

The proof goes by a Zorn's lemma argument considering all possible extensions with the given property. One shows that this is a partially ordered set and any extension which is not defined on the whole space has a nontrivial extension.

This has one corollary that we will need later.

**Corollary 16.6.** *Let $X$ be a Banach space and $F \subset B$ a finite dimensional subspace. Then $F$ has closed complementary subspace. (i.e., there is a closed subspace $C \subset B$ so that $F \cap C = \{0\}$ and $F + C = B$.*

Proof. Take a basis $\{f_1, \ldots, f)n\}$ for $F$. Let $\phi_1, \ldots, \phi_n$ be the corresponding dual basis of $F^*$. Clearly the $\phi_i$ satisfy the hypothesis of the Hahn-Banach theorem with $p$ being a multiple of the norm. So there are linear functionals $\tilde{\phi}_1, \ldots, \tilde{\phi}_n$ extending these. Set $C = \cap_{i=1}^n \ker(\tilde{\phi}_i)$.

**Theorem 16.7. The Open mapping theorem** *Any surjective bounded linear mapping $T : X \to Y$ is an open mapping, that is it takes open sets to open sets.*

The proof of this theorem is an application of the Baire category theorem.

An important corollary is the Banach isomorphism theorem.

**Theorem 16.8. The Banach isomorphism theorem** *A bounded linear map $T : X \to Y$ which is an isomorphism of vector spaces is a topological isomorphism.*

*Proof.* At issue is show that $T^{-1}$ which exists as a map of sets is continuous. So we must show for all $U \subset X$ open that $(T^{-1})^{-1}(U) = T(U)$ is open. $T$ is surjective so this following from the open mapping theorem. □

**Theorem 16.9. The closed graph theorem** *A linear operator $T : X \to Y$ is bounded if and only if its graph $\Gamma_T = \{(x, Tx) | x \in X\| \subset X \times Y$ is closed.*

## 16.2 Compact operators

In this subsubsection $X$ and $Y$ will denote Banach spaces.

**Definition 16.10.** A linear operator $T : X \to Y$ is called a compact operator the image under $T$ of the unit ball in $X$ has compact closure in $Y$.

*Remark* 2. Compact operators are sometime called completely continuous.

The prototypical compact operator is the following Let $X$ and $Y$ be the space $\ell^2$ of all sequences $a = (a_1, a_2, \ldots)$ so that $\sum_{i=1}^{\infty} (a_I)^2 \leq \infty$ and define

$$T(a_1, a_2, \ldots) = (a_1, a_2/2, a_3/3, \ldots, a_n/n, \ldots)$$

To see that $T$ is compact choose a sequence $a^i$ in $B_1$ the ball of radius one. By a diagonal argument we can pass to a subsequence where components of $a^i$ converge to some $a^{\infty}$. Then we claim that $T(a^i)$ converges in $\ell^2$. Choose $\epsilon > 0$. Then choose $i_0 > 0$ so that the following hold.

1. $\frac{1}{i_0} < \epsilon/2$

2. $(\sum_{n=1}^{i_0-1} |a_n^i - a_n^\infty|^2)^{\frac{1}{2}} \leq \epsilon/2.$

The last follows from the component-wise convergence. Then we have for $i \geq i_0$

$$\begin{aligned}
\|T(a^i) - T(a^\infty)\|^2 &\leq \sum_{n=1}^{i_0-1} \frac{1}{i^2}|a_n^i - a_n^\infty|^2 + (\sum_{n=i_0}^{\infty} \frac{1}{i^2}|a_n^i - (a_n^\infty)^2|)^{\frac{1}{2}} \\
&\leq \epsilon^2/4 + \frac{1}{i_0^2} \sum_{n=i_0}^{\infty} |a_n^i - a_n^\infty|^2 \\
&\leq \epsilon^2/4 + \epsilon^2/4 = \epsilon^2/2.
\end{aligned}$$

The basic result that we will need is Arzela-Ascoli theorem. Let $B$ be a ball in $\mathbb{R}^n$. Recall we call a subset $A \in C^0(B)$ equicontinuous if for all $\epsilon > 0$ there is a $\delta > 0$ so that if $|x - y| < \delta$ then $|f(x) - f(y)| < \epsilon$ for all $f \in A$.

**Theorem 16.11.** *(Arzela-Ascoli). A subset $A \in C^0(B)$ has compact closure in $C^0(B)$ if and only if $A$ is bounded and equicontinuous.*

This has an immediate corollary:

**Corollary 16.12.** *The embedding $C^1(B) \to C^0(B)$ is compact.*

*Proof.* The unit ball in $C^{0,\alpha}(B)$ is certainly bounded in $C^0(B)$. If $\|f\|_{C^{0,\alpha}} \leq 1$ then $|f(x) - f(y)| \leq |x - y|$ we can take $\delta = \epsilon$. $\square$