18.917 Topics in Algebraic Topology: The Sullivan Conjecture
Fall 2007

The Arithmetic Square (Lecture 32)

Our goal in this lecture is to address the following question: given a nice space X, to what extent can X be recovered from its completions at all primes? We begin by reviewing the situation for abelian groups.

Let A be a finitely generated abelian group. For each prime p, let A_p denote the p-adic completion $A \otimes_{\mathbf{Z}} \mathbf{Z}_p$. Let $A_{\mathbf{Q}}$ denote the rationalization $A \otimes_{\mathbf{Z}} \mathbf{Q}$. We have canonical maps

$$A_{\mathbf{Q}} \leftarrow A \rightarrow \prod_p A_p,$$

which fit into a commutative diagram

$$\begin{array}{ccc} A & \longrightarrow & \prod_p A_p \\ \downarrow & & \downarrow \\ A_{\mathbf{Q}} & \longrightarrow & (\times_p A_p)_{\mathbf{Q}}. \end{array}$$

Remark 1. This diagram is a pullback square: in other words, it determines a short exact sequence

$$0 \rightarrow A \rightarrow A_{\mathbf{Q}} \times \prod_p A_p \rightarrow (\times_p A_p)_{\mathbf{Q}} \rightarrow 0.$$

We wish to prove an analogue of this result where the abelian group A is replaced by a nice topological space X.

We first discuss the rationalization of topological spaces.

Definition 2. Let $f : X \rightarrow Y$ be a map of topological spaces. We say that f is a *rational homotopy equivalence* if it induces an isomorphism on rational cohomology $\mathrm{H}^*(Y; \mathbf{Q}) \rightarrow \mathrm{H}^*(X; \mathbf{Q})$ (this is equivalent to the assertion that f induces an isomorphism on rational homology). We say that a space Z is *rational* (or $\mathbf{Q}$-local) if, for every rational homotopy equivalence $f : X \rightarrow Y$, the induced map

$$\mathrm{Map}(Y, Z) \rightarrow \mathrm{Map}(X, Z)$$

is a homotopy equivalence.

Given a topological space X, a *rationalization* of X is a topological space X' equipped with a rational homotopy equivalence $X \rightarrow X'$, such that X' is rational.

If X is any topological space, then a rationalization X' of X is determined by X, up to canonical homotopy equivalence. This follows from Yoneda's lemma: for any rational space Z, we have an equivalence of mapping spaces $\mathrm{Map}(X', Z) \simeq \mathrm{Map}(X, Z)$, so that the functor co-represented by X' (on rational spaces) is already determined by X. A fundamental result of Bousfield implies that every space X admits a rationalization. We will be content to prove the following less general, but more explicit result:

Theorem 3. *Let X be a simply connected topological space. Then:*

(1) *A map $X \to X'$ is a rationalization of X if and only if X' is simply connected, and for each $i > 1$ the map $\pi_i X \to \pi_i X'$ induces an isomorphism $\pi_i X \otimes_{\mathbf{Z}} \pi_i X'$.*

(2) *X admits a rationalization $X_{\mathbf{Q}}$.*

The proof proceeds in several steps.

Lemma 4. *Let Z be a simply connected topological space. Assume that each homotopy group $\pi_i Z$ is a vector space over the rational numbers. Then Z is rational.*

Remark 5. The converse is also true; this follows from Theorem 3.

Proof. Suppose first that Z is an Eilenberg-MacLane space $K(V, n)$, where V is a rational vector space. Then, for any space X, we have

$$\pi_i \operatorname{Map}(X, Z) \simeq \mathrm{H}^{n-i}(X; V).$$

If $f : X \to Y$ is a rational equivalence, then f induces an isomorphism on rational homology. It follows from the universal coefficient theorem that f induces an isomorphism on cohomology with coefficients in V, so that f induces a homotopy equivalence $\operatorname{Map}(Y, Z) \to \operatorname{Map}(X, Z)$. This proves that Z is rational.

We now consider the general case. The space Z is the homotopy limit of its Postnikov tower

$$\ldots \to \tau_{\leq n} Z \to \ldots \tau_{\leq 1} Z \simeq *.$$

Since the collection of rational spaces is stable under homotopy limits, it will suffice to show that each $\tau_{\leq n} Z$ is rational. The proof proceeds by induction on n. We have a homotopy pullback diagram

$$\begin{array}{ccc} \tau_{\leq n} Z & \longrightarrow & * \\ \downarrow & & \downarrow \\ \tau_{\leq n-1} Z & \longrightarrow & K(\pi_n Z, n+1). \end{array}$$

The inductive hypothesis implies that $\tau_{\leq n-1} Z$ is rational, and the first part of the proof shows that $K(\pi_n Z, n+1)$ is rational. It follows that $\tau_{\leq n} Z$ is also rational, as desired. □

We now prove the "if" direction of assertion (1) in Theorem 3. Let $f : X \to X'$ be a map of simply connected spaces which induces isomorphisms $\pi_i X \otimes_{\mathbf{Z}} \mathbf{Q} \to \pi_i X'$ for $i > 1$. We wish to show that X' is a rationalization of X. Lemma 4 shows that X' is rational; it therefore suffices to show that f induces an isomorphism on rational cohomology. We have a fiber sequence

$$F \to X \to X'.$$

In view of the Serre spectral sequence, it suffices to show that the rational cohomology of F is trivial in positive degrees. The long exact sequence of homotopy groups shows that the homotopy groups of F consist entirely of torsion. The desired result is therefore an immediate consequence of the following:

Lemma 6. *Let F be a connected space, and assume that the homotopy groups of F are abelian torsion groups. Then $\mathrm{H}_*(F; \mathbf{Q})$ vanishes for $* > 0$.*

Proof. We will prove by induction on i that the statement holds for the Postnikov section $\tau_{\leq i} F$. Since $\mathrm{H}_i(F; \mathbf{Q}) \simeq \mathrm{H}_i(\tau_{\leq i} F; \mathbf{Q})$, this will imply the desired result. Using the inductive hypothesis and the Serre spectral sequence, we can reduce to the case where F is an Eilenberg-MacLane space $K(A, i)$, where A is an abelian torsion group. Then A is a filtered colimit of finite abelian groups; we may therefore reduce to the case where A is finite. Using the Eilenberg-Moore spectral sequence, we can reduce to the case where $i = 1$. We now appeal to the following fact: in positive degrees, the homology groups of a finite group A are annihilated by the order $|A|$; in particular, the rational homology groups vanish. □

We now prove the following version of the second part of Theorem 3:

(2′) Let X be a simply connected topological space. Then there exists a map $f : X \to X_{\mathbf{Q}}$, where $X_{\mathbf{Q}}$ is simply connected and f induces isomorphisms $\pi_i X \otimes_{\mathbf{Z}} \mathbf{Q} \to \pi_i X_{\mathbf{Q}}$.

In view of what we have proven above, the space $X_{\mathbf{Q}}$ will automatically be a rationalization of X, and therefore functorially determined by X.

We now prove (2′) under the additional assumption that the homotopy groups $\pi_i X$ vanish for $i > n$, using induction on n. If $n = 1$, then X is contractible and there is nothing to prove. In general, if we let τX denote the space obtained by killing the nth homotopy group of X, then we have a homotopy pullback diagram

$$\begin{array}{ccc} X & \longrightarrow & * \\ \downarrow & & \downarrow \\ \tau X & \longrightarrow & K(\pi_n X, n+1). \end{array}$$

Using the inductive hypothesis and the first step, we can extend this diagram as follows:

$$\begin{array}{ccc} X & \longrightarrow & * \\ \downarrow & & \downarrow \\ \tau X & \longrightarrow & K(\pi_n X, n+1) \\ \downarrow & & \downarrow \\ (\tau X)_{\mathbf{Q}} & \longrightarrow & K(\pi_n X \otimes_{\mathbf{Z}} \mathbf{Q}, n+1). \end{array}$$

Here we have invoked the fact that $(\tau X)_{\mathbf{Q}}$ is a rationalization of τX to complete the bottom square. The outer square determines a map from X into the homotopy pullback

$$X_{\mathbf{Q}} = (\tau X)_{\mathbf{Q}} \times_{K(\pi_n X \otimes_{\mathbf{Z}} \mathbf{Q}, n+1)} *.$$

It is easily checked that $X_{\mathbf{Q}}$ has the desired properties.

We now handle the general case. The simply connected space admits a Postnikov tower

$$\ldots \to \tau_{\leq n} X \to \tau_{\leq n-1} X \to \ldots \to \tau_{\leq 1} X \simeq *.$$

Since the process of rationalization is functorial and (2′) is satisfied by each $\tau_{\leq k} X$, we get an induced tower

$$\ldots \to (\tau_{\leq n} X)_{\mathbf{Q}} \to \ldots \to (\tau_{\leq 1} X)_{\mathbf{Q}} \simeq *.$$

Let $X_{\mathbf{Q}}$ denote the homotopy inverse limit of this tower; it is easy to see that $X_{\mathbf{Q}}$ has the desired properties. This completes the proof of (2′), and therefore the proof of part (2) of Theorem 3.

We now prove the "only if" direction of Theorem 3. Let X be a simply connected topological space. In view of (2′), there exists a rationalization $X \to X_{\mathbf{Q}}$ which induces isomorphisms $\pi_i X \otimes_{\mathbf{Z}} \mathbf{Q} \to \pi_i X_{\mathbf{Q}}$. Since a rationalization of X is determined up to homotopy equivalence by X, it follows that any rationalization of X has the same property.

We are now ready to return to the main theme of this lecture. Let X be a simply connected topological space, and assume that each homotopy group $\pi_i X$ is finitely generated. For every prime p, let $\widehat{X}_p = \varprojlim X_p^{\vee}$ denote the p-adic completion of X discussed in the last lecture. We have a canonical map

$$X \to \prod_p \widehat{X}_p.$$

Both sides are simply connected, and therefore admit rationalizations. We get a homotopy commutative diagram

$$\begin{array}{ccc} X & \longrightarrow & (\prod_p \widehat{X}_p) \\ \downarrow & & \downarrow \\ X_{\mathbf{Q}} & \longrightarrow & (\prod_p \widehat{X}_p)_{\mathbf{Q}}. \end{array}$$

Theorem 7. *Let X be a simply connected space whose homotopy groups are finitely generated. Then the preceding diagram is a homotopy pullback square.*

In other words, under reasonable connectedness and finiteness assumptions, any space X can be recovered by "gluing" together its rationalizations and its completions at all primes.

Proof. Let Y denote the homotopy fiber product

$$(\prod_p \widehat{X}_p) \times_{(\prod_p \widehat{X}_p)_{\mathbf{Q}}} X_{\mathbf{Q}},$$

so that we have a canonical map $\alpha : X \to Y$ and we wish to show that it is a homotopy equivalence. By construction, the homotopy groups of Y fit into a long exact sequence

$$\ldots \to \pi_n Y \to \pi_n X_{\mathbf{Q}} \times \pi_n(\prod_p \widehat{X}_p) \stackrel{\phi_n}{\to} \pi_n(\prod_p \widehat{X}_p)_{\mathbf{Q}} \to \ldots$$

Let $A = \pi_n X$. Then we can identify the domain of ϕ_n with the product $A_{\mathbf{Q}} \times \prod_p A_p$, and the codomain of ϕ_n with $(\prod_p A_p)_{\mathbf{Q}}$. Remark 1 implies that ϕ_n is surjective. It follows that the long exact sequences above breaks up into short exact sequences, and gives isomorphisms

$$\pi_n Y \simeq \ker(\phi_n) \simeq A.$$

These isomorphisms are induced by the map $A \to \pi_n X \to \pi_n Y$, so that α is a homotopy equivalence as desired. □