

9 The homology long exact sequence

A pair of spaces (X, A) gives rise to a short exact sequence of chain complexes:

$$0 \rightarrow S_*(A) \rightarrow S_*(X) \rightarrow S_*(X, A) \rightarrow 0.$$

In homology, this will relate $H_*(A)$, $H_*(X)$, and $H_*(X, A)$.

To investigate what happens, let's suppose we have a general short exact sequence of chain complexes,

$$0 \rightarrow A_* \rightarrow B_* \rightarrow C_* \rightarrow 0,$$

and study what happens in homology. Clearly the composite $H_*(A) \rightarrow H_*(B) \rightarrow H_*(C)$ is trivial. Is this sequence exact? Let $[b] \in H_n(B)$ such that $g([b]) = 0$. It's determined by some $b \in B_n$ such that $d(b) = 0$. If $g([b]) = 0$, then there is some $\bar{c} \in C_{n+1}$ such that $d\bar{c} = gb$. Now, g is surjective, so there is some $\bar{b} \in B_{n+1}$ such that $g(\bar{b}) = \bar{c}$. Then we can consider $d\bar{b} \in B_n$, and $g(d\bar{b}) = d(\bar{c}) \in C_n$. What is $b - d\bar{b}$? This maps to zero in C_n , so by exactness there is some $a \in A_n$ such that $f(a) = b - d\bar{b}$. Is a a cycle? Well, $f(da) = d(fa) = d(b - d\bar{b}) = db - d^2\bar{b} = db$, but we assumed that $db = 0$, so $f(da) = 0$. This means that da is zero because f is an injection by

exactness. Therefore a is a cycle. What is $[a] \in H_n(A)$? Well, $f([a]) = [b - d\bar{b}] = [b]$. This proves exactness of $H_n(A) \rightarrow H_n(B) \rightarrow H_n(C)$.

On the other hand, $H_*(A) \rightarrow H_*(B)$ may fail to be injective, and $H_*(B) \rightarrow H_*(C)$ may fail to be surjective. Instead:

Theorem 9.1 (The homology long exact sequence). *Let $0 \rightarrow A_* \rightarrow B_* \rightarrow C_* \rightarrow 0$ be a short exact sequence of chain complexes. Then there is a natural homomorphism $\partial : H_n(C) \rightarrow H_{n-1}(A)$ such that the sequence*

$$\begin{array}{ccccccc} & & & \cdots & \longrightarrow & H_{n+1}(C) & \\ & & & & \searrow & \partial & \\ & & & & & & \\ H_n(A) & \longrightarrow & H_n(B) & \longrightarrow & H_n(C) & & \\ & & & & \searrow & \partial & \\ & & & & & & \\ H_{n-1}(A) & \longrightarrow & \cdots & & & & \end{array}$$

is exact.

Proof. We'll construct ∂ , and leave the rest as an exercise. Here's an expanded part of this short exact sequence:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A_{n+1} & \xrightarrow{f} & B_{n+1} & \xrightarrow{g} & C_{n+1} & \longrightarrow & 0 \\ & & \downarrow d & & \downarrow d & & \downarrow d & & \\ 0 & \longrightarrow & A_n & \xrightarrow{f} & B_n & \xrightarrow{g} & C_n & \longrightarrow & 0 \\ & & \downarrow d & & \downarrow d & & \downarrow d & & \\ 0 & \longrightarrow & A_{n-1} & \xrightarrow{f} & B_{n-1} & \xrightarrow{g} & C_{n-1} & \longrightarrow & 0 \end{array}$$

Let $c \in C_n$ be a cycle: $dc = 0$. The map g is surjective, so pick a $b \in B_n$ such that $g(b) = c$, and consider $db \in B_{n-1}$. Well, $g(db) = d(g(b)) = dc = 0$. So by exactness, there is some $a \in A_{n-1}$ such that $f(a) = db$. How many choices are there of picking a ? Only one, because f is injective. We need to check that a is a cycle. What is $d(a)$? Well, $d^2b = 0$, so da maps to 0 under f . But because f is injective, $da = 0$, i.e., a is a cycle. This means we can define $\partial[c] = [a]$.

To make sure that this is well-defined, let's make sure that this choice of homology class a didn't depend on the b that we chose. Pick some other b' such that $g(b') = c$. Then there is $a' \in A_{n-1}$ such that $f(a') = db'$. We want $a - a'$ to be a boundary, so that $[a] = [a']$. We want $\bar{a} \in A_n$ such that $d\bar{a} = a - a'$. Well, $g(b - b') = 0$, so by exactness, there is $\bar{a} \in A_n$ such that $f(\bar{a}) = b - b'$. What is $d\bar{a}$? Well, $d\bar{a} = d(b - b') = db - db'$. But $f(a - a') = b - b'$, so because f is injective, $d\bar{a} = a - a'$, i.e., $[a] = [a']$. I leave the rest of what needs checking to the listener. \square

Example 9.2. A pair of spaces (X, A) gives rise to a natural long exact sequence in homology:

$$\begin{array}{ccccccc} & & & \cdots & \longrightarrow & H_{n+1}(X, A) & . \\ & & & & \searrow & \partial & \\ & & & & & & \\ H_n(A) & \longrightarrow & H_n(X) & \longrightarrow & H_n(X, A) & & \\ & & & & \searrow & \partial & \\ & & & & & & \\ H_{n-1}(A) & \longrightarrow & \cdots & & & & \end{array}$$

Example 9.3. Let's think again about the pair (D^n, S^{n-1}) . By homotopy invariance we know that $H_q(D^n) = 0$ for $q > 0$, since D^n is contractible. So

$$\partial : H_q(D^n, S^{n-1}) \rightarrow H_{q-1}(S^{n-1})$$

is an isomorphism for $q > 1$. The bottom of the long exact sequence looks like this:

$$\begin{array}{ccccccc} & & & & 0 & \longrightarrow & H_1(D^n, S^{n-1}) \\ & & & & & \swarrow & \\ H_0(S^{n-1}) & \longrightarrow & H_0(D^n) & \longrightarrow & H_0(D^n, S^{n-1}) & \longrightarrow & 0 \end{array}$$

When $n > 1$, both S^{n-1} and D^n are path-connected, so the map $H_0(S^{n-1}) \rightarrow H_0(D^n)$ is an isomorphism, and

$$H_1(D^n, S^{n-1}) = H_0(D^n, S^{n-1}) = 0.$$

When $n = 1$, we discover that

$$H_1(D^1, S^0) = \mathbf{Z} \quad \text{and} \quad H_0(D^1, S^0) = 0.$$

The generator of $H_1(D^1, S^0)$ is represented by any 1-simplex $\iota_1 : \Delta^1 \rightarrow D^1$ such that $d_0\iota = c_1^0$ and $d_1\iota = c_0^0$ (or vice versa). To go any further in this analysis, we'll need another tool, known as "excision."

We can set this up for reduced homology (as in Lecture 5) as well. Note that any map induces an isomorphism in \tilde{S}_{-1} , so to a pair (X, A) we can associate a short exact sequence

$$0 \rightarrow \tilde{S}_*(A) \rightarrow \tilde{S}_*(X) \rightarrow S_*(X, A) \rightarrow 0$$

and hence a long exact sequence

$$\begin{array}{ccccccc} & & & & \cdots & \longrightarrow & H_{n+1}(X, A) \ . \\ & & & & & \swarrow & \partial \\ \tilde{H}_n(A) & \longrightarrow & \tilde{H}_n(X) & \longrightarrow & H_n(X, A) & & \\ & & & & & \swarrow & \partial \\ \tilde{H}_{n-1}(A) & \longrightarrow & \cdots & & & & \end{array}$$

In the example (D^n, S^{n-1}) , $\tilde{H}_*(D^n) = 0$ and so

$$\partial : H_q(D^n, S^{n-1}) \xrightarrow{\cong} \tilde{H}_{q-1}(S^{n-1})$$

for all n and q . This even works when $n = 0$; remember that $S^{-1} = \emptyset$ and $\tilde{H}_{-1}(\emptyset) = \mathbf{Z}$. This is why I like this convention.

The homology long exact sequence is often used in conjunction with an elementary fact about a map between exact sequences known as the *five lemma*. Suppose you have two exact sequences of abelian groups and a map between them – a "ladder":

$$\begin{array}{ccccccccc} A_4 & \xrightarrow{d} & A_3 & \xrightarrow{d} & A_2 & \xrightarrow{d} & A_1 & \xrightarrow{d} & A_0 \\ \downarrow f_4 & & \downarrow f_3 & & \downarrow f_2 & & \downarrow f_1 & & \downarrow f_0 \\ B_4 & \xrightarrow{d} & B_3 & \xrightarrow{d} & B_2 & \xrightarrow{d} & B_1 & \xrightarrow{d} & B_0 \end{array}$$

When can we guarantee that the middle map f_2 is an isomorphism? We're going to "diagram chase." Just follow your nose, making assumptions as necessary.

Surjectivity: Let $b_2 \in B_2$. We want to show that there is something in A_2 mapping to b_2 . We can consider $db_2 \in B_1$. Let's assume that f_1 is surjective. Then there's $a_1 \in A_1$ such that $f_1(a_1) = db_2$. What is da_1 ? Well, $f_0(da_1) = d(f_1(a_1)) = d(db_2) = 0$. So we want f_0 to be injective. Then da_1 is zero, so by exactness of the top sequence, there is some $a_2 \in A_2$ such that $da_2 = a_1$. What is $f_2(a_2)$? To answer this, begin by asking: What is $d(f_2(a_2))$? By commutativity, $d(f_2(a_2)) = f_1(d(a_2)) = f_1(a_1) = db_2$. Let's consider $b_2 - f_2(a_2)$. This maps to zero under d . So by exactness, there is $b_3 \in B_3$ such that $d(b_3) = b_2 - f_2(a_2)$. If we assume that f_3 is surjective, then there is $a_3 \in A_3$ such that $f_3(a_3) = b_3$. But now $d(a_3) \in A_2$, and $f_2(d(a_3)) = d(f_3(a_3)) = b_2 - f_2(a_2)$. This means that $b_2 = f_2(a_2 + d(a_3))$, verifying surjectivity of f_2 .

This proves the first half of the following important fact. The second half is "dual" to the first.

Proposition 9.4 (Five lemma). *In the map of exact sequences above,*

- *If f_0 is injective and f_1 and f_3 are surjective, then f_2 is surjective.*
- *If f_4 is surjective and f_3 and f_1 are injective, then f_2 is injective.*

Very commonly one knows that f_0, f_1, f_3 , and f_4 are all isomorphisms, and concludes that f_2 is also an isomorphism. For example:

Corollary 9.5. *Let*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A'_* & \longrightarrow & B'_* & \longrightarrow & C'_* & \longrightarrow & 0 \\ & & \downarrow f & & \downarrow g & & \downarrow h & & \\ 0 & \longrightarrow & A_* & \longrightarrow & B_* & \longrightarrow & C_* & \longrightarrow & 0 \end{array}$$

be a map of short exact sequences of chain complexes. If two of the three maps induced in homology by f, g , and h are isomorphisms, then so is the third.

Here's an application.

Proposition 9.6. *Let $(A, X) \rightarrow (B, Y)$ be a map of pairs, and assume that two of $A \rightarrow B$, $X \rightarrow Y$, and $(X, A) \rightarrow (Y, B)$ induce isomorphisms in homology. Then the third one does as well.*

Proof. Just apply the five lemma to the map between the two homology long exact sequences. \square

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