18.905: Problem Set I

Homework is an important part of this class. I hope you gain from the struggle. Collaboration can be effective, but be sure that you grapple with each problem on your own as well. If you do work with others, you must indicate with whom on your solution sheet. Scores will be posted on the Stellar website.

1. (a) Let $[n]$ denote the totally ordered set $\{0, 1, \ldots, n\}$. Let $\phi : [m] \to [n]$ be an order preserving function (so that if $i \leq j$ then $\phi(i) \leq \phi(j)$). Identifying the elements of $[n]$ with the vertices of the standard simplex Δ^n, ϕ extends to an affine map $\Delta^m \to \Delta^n$ that we also denote by ϕ. Give a formula for this map in terms of barycentric coordinates: If $\phi(s_0, \ldots, s_m) = (t_0, \ldots, t_n)$, what is t_j as a function of $(s_0, \ldots, s_m)$?

(b) Write $d^j : [n-1] \to [n]$ for the order preserving injection that omits j as a value. Show that an order preserving injection $\phi : [n-k] \to [n]$ is uniquely a composition of the form $d^{j_k} d^{j_{k-1}} \cdots d^{j_1}$, with $0 \leq j_1 < j_2 < \cdots < j_k \leq n$. Do this by describing the integers $j_1, \ldots, j_k$ directly in terms of ϕ, and then verify the straightening rule

$$d^i d^j = d^{j+1} d^i \quad \text{for} \quad i \leq j$$

(c) Show that any order preserving map $\phi : [m] \to [n]$ factors uniquely as the composition of an order preserving surjection followed by an order preserving injection

(d) Write $s^i : [m+1] \to [m]$ for the order-preserving surjection that repeats the value i. Show that any order-preserving surjection $\phi : [m] \to [n]$ has a unique expression $(s^n)^{i_n} (s^{n-1})^{i_{n-1}} \cdots (s^0)^{i_0}$. Do this by describing the numbers $i_0, \ldots, i_n$, directly in terms of ϕ, and finding a straightening rule of the form $s^i s^j = \cdots$ for $i < j$.

(e) Finally, implement your assertion that any order preserving map factors as a surjection followed by an injection by establishing a straightening rule of the form $s^i d^j = \cdots$.

Recall the notation $\mathrm{Sin}_n(X)$ for the set of continuous maps from Δ^n to the space X. The affine extension $\phi : \Delta^m \to \Delta^n$ of an order-preserving map $\phi : [m] \to [m]$ induces a map $\phi^* : \mathrm{Sin}_n(X) \to \mathrm{Sin}_m(X)$. In particular, write

$$d_i = (d^i)^* \qquad s_j = (s^j)^* .$$

The d_i's are *face maps*, the s_i's are *degeneracies.*

(f) Write down the identities satisfied by these operators, resulting from the identities you found relating the d^i's and s^j's.

A *simplicial set* is a sequence of sets $K_0, K_1, \ldots$, with maps $d_i : K_n \to K_{n-1}$, $0 \leq i \leq n$, and $s_i : K_n \to K_{n+1}$, $0 \leq i \leq n$, satisfying these identities. For example, we have the *singular simplicial set* $\mathrm{Sin}_*(X)$ of a space X.

The group of n-chains on X is the free abelian group generated by the n-simplices:

$$S_n(X) = \mathbb{Z}\mathrm{Sin}_n(X) .$$

Each operator $\phi : \mathrm{Sin}_n(X) \to \mathrm{Sin}_m(X)$ extends uniquely to a homomorphism $\phi : S_n(X) \to S_m(X)$. In particular, $d_i : S_n(X) \to S_{n-1}(X)$, $0 \le i \le n$. We can combine these homomorphism to form the *boundary map*

$$d = \sum_{i=0}^{n} (-1)^i d_i : S_n(X) \to S_{n-1}(X)\,.$$

(g) Use the relations among the d_i's to prove that

$$d^2 = 0 : S_n(X) \to S_{n-2}(X)\,.$$

(h) Let $f : X \to Y$ be a continuous map. By composition, we get a map $f_* : \mathrm{Sin}_n(X) \to \mathrm{Sin}_n(Y)$. Show that together these form a map of simplicial sets (i.e. they commute with the maps induced by order preserving maps $\phi : [m] \to [n]$).

2. **(a)** Write down a singular 2-cycle representing the "fundamental class" of the torus $T^2 = S^1 \times S^1$. We will give a precise definition of the fundamental class of a manifold later, but for now let's just say that this cycle should be made up of singular 2-simplices which together cover all but a small (e.g. nowhere dense) subset of T^2 exactly once.

(b) Construct an isomorphism

$$H_n(X) \oplus H_n(Y) \to H_n(X \amalg Y)\,.$$

3. **(a)** Write $\pi_0(X)$ for the set of path-components of a space X. Construct an isomorphism

$$\mathbb{Z}\pi_0(X) \to H_0(X)\,.$$

(b) Say what it means to assert that the isomorphisms you constructed in **2(b)** and **3(a)** are *natural*, and make sure they are.

4. Here are a couple more "categorical" definitions, giving you some practice with the idea of constructions being defined by universal mapping properties.

Let $\mathbf{C}$ be a category, A a set, and $a \mapsto X_a$ an assignment of an object of $\mathbf{C}$ to each element of A. A *product* of these objects is an object Y together with maps $\mathrm{pr}_a : Y \to X_a$ with the following property. For any object Z and any family of maps $f_a : Z \to X_a$, there is a unique map $Z \to Y$ such that $f_a = \mathrm{pr}_a \circ f$ for all $a \in A$. A *coproduct* of these objects is an object Y together with maps $\mathrm{in}_a : X_a \to Y$ with the following property. For any object Z and any family of maps $f_a : X_a \to Z$, there is a unique map $Y \to Z$ such that $f_a = f \circ \mathrm{in}_a$ for all $a \in A$.

(a) Describe constructions of the product and coproduct (if they exist) in the following categories: sets, pointed sets, spaces, abelian groups. (A *pointed set* is a pair $(S, *)$ where S is a set and $* \in S$.)

(b) What should be meant by the product when $A = \varnothing$? How about the coproduct? What are these objects in the four categories mentioned in **(a)**? Give an example of a category in which neither one of these constructions exists.

(c) Show that if $(Y, \{\mathrm{pr}_a\})$ and $(Y', \{\mathrm{pr}'_a\})$ are both products of a family $\{X_a : a \in A\}$, then there is a unique map $f : Y \to Y'$ such that $pr'_a \circ f = \mathrm{pr}_a$ for all $a \in A$, and that this map is an isomorphism.

(d) A partially ordered set S defines a category as follows. The objects are the elements of S. (They constitute a *set*, rather than something larger, so this is a "small category.") For $s, s' \in S$, there is exactly one morphism $s \to s'$ if $s \le s'$, and none otherwise.

Take S to be the real numbers $\mathbf{R}$ with their natural order, for example, and consider a map $A \to \mathbf{R}$. Under what conditions does the product of these objects exist, and if it does what is it? Same question for the coproduct.

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