Tychonoff via well-ordering.

We present a proof of the Tychonoff theorem that uses the well-ordering theorem rather than Zorn's lemma. It follows the outline of Exercise 5 of §37.

Lemma H.1. Let $\mathcal{A}$ be a collection of basis elements for the topology of the product space $X \times Y$, such that no finite subcollection of $\mathcal{A}$ covers $X \times Y$. If X is compact, there is a point $x \in X$ such that no finite subcollection of $\mathcal{A}$ covers the slice $\{x\} \times Y$.

Proof. Suppose there is no such point x. Then, given a point x of X, one can choose finitely many elements of $\mathcal{A}$ that cover the slice $\{x\} \times Y$. Then, as in the proof of the tube lemma, one can find a neighborhood U_x of x such that these elements of $\mathcal{A}$ cover $U_x \times Y$. Because X is compact, we can cover X by finitely many such neighborhoods U_x; then all of $X \times Y$ can be covered by finitely many elements of $\mathcal{A}$. $\square$

Theorem H.2. Products of compact spaces are compact.

Proof. Let $\{X_\alpha\}_{\alpha \in J}$ be a family of compact spaces; let X be their product,

$$X = \prod_{\alpha \in J} X_\alpha ;$$

and let $\pi_\alpha : X \to X_\alpha$ be the projection map. Well-order J in such a way that it has a largest element.

Step 1. Let β be an element of J; and suppose that a point p_i of X_i has been specified for all $i < \beta$. Define Z_β to be the following subspace of X:

$$Z_\beta = \{\underline{x} \mid \pi_i(\underline{x}) = p_i \text{ for } i < \beta\} .$$

Then for each $\alpha < \beta$, define Y_α to be the following subspace of X:

$$Y_\alpha = \{\underline{x} \mid \pi_i(\underline{x}) = p_i \text{ for } i \leq \alpha\} .$$

Note that as α increases, the space Y_α shrinks, and that Z_β equals the intersection of the spaces Y_α for all $\alpha < \beta$.

We show that if $\mathcal{A}$ is a finite collection of basis elements for X that covers Z_β, then $\mathcal{A}$ actually covers the larger space Y_α, for some $\alpha < \beta$.

If β has an immediate predecessor in J, let α be that immediate predecessor. Then $Y_\alpha = Z_\beta$, and the result is trivial.

Now suppose that β has no immediate predecessor. For each element A of $\mathcal{A}$, let J_A denote the set of those indices $i < \beta$ for which $\pi_i(A) \neq X_i$; then J_A is a finite set. The union of the sets J_A, for all A in $\mathcal{A}$, is also finite; let α be the largest element of this union. Then $\alpha < \beta$, and $\pi_i(A) = X_i$ whenever i is an index such that $\alpha < i < \beta$ and A is an element of $\mathcal{A}$.

We show that $\mathcal{A}$ covers Y_α. Given $\underline{x} \in Y_\alpha$, we show that it lies in an element of $\mathcal{A}$. We know that $\pi_i(\underline{x}) = p_i$ for $i \leq \alpha$. Define a point $\underline{y}$ of X by setting

$$\pi_i(\underline{y}) = p_i \quad \text{for } i < \beta, \text{ and}$$
$$\pi_i(\underline{y}) = \pi_i(\underline{x}) \quad \text{for } i \geq \beta .$$

Then $\underline{y}$ belongs to Z_β, so that $\underline{y}$ lies in some element A of $\mathcal{A}$. We show this element of $\mathcal{A}$ also contains $\underline{x}$.

Since A is a basis element, we need only to show that $\pi_i(\underline{x}) \in \pi_i(A)$ for all $i \in J$. Since $\underline{y} \in A$, we know that $\pi_i(\underline{y}) \in \pi_i(A)$ for all i. We also know that $\pi_i(\underline{x}) = \pi_i(\underline{y})$ for $i \leq \alpha$ and for $i \geq \beta$. And finally, for $\alpha < i < \beta$ we know that $\pi_i(\underline{x}) \in \pi_i(A)$ because in this case $\pi_i(A) = X_i$.

Step 2. Assume that $\mathcal{A}$ is a collection of basis elements for X such that no finite subcollection covers X. We show that $\mathcal{A}$ itself does not cover X. The theorem follows.

We shall choose points $p_i \in X_i$, for all i, such that none of the spaces Y_α, for $\alpha \in J$, can be finitely covered by $\mathcal{A}$. When α is the largest element of J, the space Y_α is a one-point space. Since it cannot be finitely covered by $\mathcal{A}$, it is not contained in any element of $\mathcal{A}$.

To begin, let α be the smallest element of J. We write X in the form

$$X_\alpha \times \prod_{i \neq \alpha} X_i \quad .$$

Since X cannot be finitely covered by $\mathcal{A}$, and since X_α is compact, the preceding lemma implies that there is a point $p_\alpha \in X_\alpha$ such that the space

$$Y_\alpha = \{p_\alpha\} \times \prod_{i \neq \alpha} X_i$$

cannot be finitely covered by $\mathcal{A}$.

Now suppose p_i is defined for all $i < \beta$, such that for each $\alpha < \beta$, the space Y_α cannot be finitely covered by $\mathcal{A}$. We seek to define the point p_β. Since none of the spaces Y_α, for $\alpha < \beta$, can be finitely covered by $\mathcal{A}$, Step 1 implies that Z_β cannot be finitely covered by $\mathcal{A}$. Let us write Z_β in the form

$$Z_\beta = \prod_{i<\beta}\{p_i\} \times X_\beta \times \prod_{i>\beta} X_i .$$

Because X_β is compact, the lemma tells us there is a point $p_\beta \in X_\beta$ such that the space

$$\prod_{i<\beta}\{p_i\} \times \{p_\beta\} \times \prod_{i>\beta} X_i$$

cannot be finitely covered by $\mathcal{A}$. This is just the space Y_β.

By the general principle of recursive definition (see p.72), p_i is defined for all i. Note of course that we have used the axiom of choice repeatedly to choose the points p_i. $\square$