Countability axioms

We have studied four basic countability properties:

(1) The first countability axiom.

(2) The second countability axiom,

(3) The Lindelöf condition.

(4) The condition that the space has a countable dense subset.

We know that condition (2) implies each of the others. We show now that this is the only general theorem relating these four conditions.

We shall in fact find, for each subset of conditions (1), (3), and (4), a space that satisfies the conditions in the subset, and none of the others. This requires seven distinct examples!

Incidentally, there do exist relations among these four conditions for certain types of spaces. For instance, for metrizable spaces, condition (1) is automatically satisfied, and the other three conditions are equivalent to one another. (See Exercise 5 of §30.) Similarly, for topological groups that are first-countable, conditions (2), (3), and (4) are equivalent. (See Exercise 18 of §30.)

Example 1. Conditions (1), (3), and (4). The space $\mathbb{R}_\ell$ is first-countable, Lindelöf, and has a countable dense subset , but is not second-countable. (See Example 3 of §30.)

Example 2. Conditions (1) and (3). The ordered square is compact, and hence Lindelöf. It is readily seen to be first-countable. It does not have a countable dense subset, since each dense subset must contain at least one point of each interval $a \times (0,1)$.

Example 3. Conditions (1) and (4). The space $\mathbb{R}_\ell \times \mathbb{R}_\ell$ is first-countable, and the rational points form a countable dense subset. It is not Lindelöf; see Example 4 of §30.

Example 4. Conditions (3) and (4). The space I^I is not first-countable; the proof given in Example 2 of §21 for $\mathbb{R}^J$ works also here. It is compact, by the Tychonoff theorem, so it is Lindelöf. We construct a countable dense subset of I^I as follows:

Given a partition

$$0 = a_0 < a_1 < \dots < a_n = 1$$

of the interval $I = [0,1]$, where the a_i are rational, and given a sequence

$$b_1, \dots, b_n$$

of rational numbers, let us define a step function $f : I \to I$ by setting

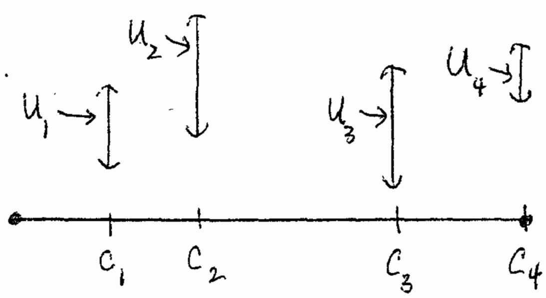
$$(*) \qquad \begin{aligned} f(x) &= b_i \quad \text{for} \quad a_{i-1} \leq x < a_i \quad (i = 1,\dots,n) \\ f(a_n) &= b_n. \end{aligned}$$

Then f is an element of I^I; and the set of all such f is countable. We shall show that these functions form a dense subset of I^I.

Let us take a typical basis element B for I^I; it is the intersection of finitely many sets of the form

$$\pi_{c_i}^{-1}(U_i) \ ,$$

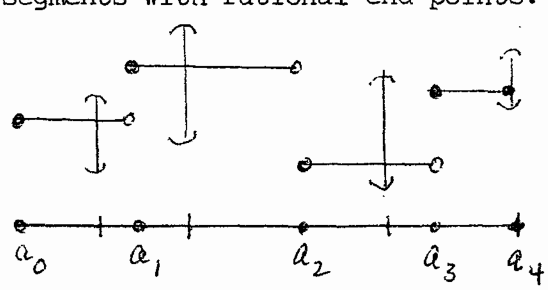
for $i = 1,\dots,n$, where $c_1 < c_2 < \dots < c_n$ are points of I and U_i is an open set of I, for each i. The set B consists of all functions from I to I whose graphs intersect the vertical intervals in the following figure.



Given B, let us choose rational numbers a_i such that

$$0 = a_0 \leq c_1 < a_1 < c_2 < a_3 < \dots < c_n \leq a_n = 1.$$

Then, for each i, choose b_i to be a rational number in the open set U_i. The corresponding function f has the graph pictured; it consists of horizontal line segments with rational end points.



Example 5. Condition (1). The space S_Ω is first-countable, but it is not Lindelöf. (Take the open cover by sets of the form S_α, for $\alpha < \Omega$.) Nor does it have a countable dense subset.

Example 6. Condition (3). The space $\bar{S}_\Omega$ is not first-countable, nor does it have a countable dense subset. But it is Lindelöf, being compact.

Example 7. Condition (4). The space $\mathbb{R}^I$ is not first-countable; see Example 2 of §21. Nor is it Lindelöf; for it is regular, and a regular Lindelöf space is normal (see Exercise 4 of §32); but $\mathbb{R}^I$ is not normal. (See Notes F.) Finally, we note that if a space has a countable dense subset, then so does any open subspace of it. The space $\mathbb{R}^I$ is homeomorphic to the space $(0,1)^I$, which is an open subspace of I^I; therefore it has a countable dense subset.