The Long Line

We follow the outline of Exercise 12 of §24.

Let L denote the set $S_\Omega \times [0,1)$, in the dictionary order. Let α_0 denote the smallest element of S_Ω. Give L the order topology.

Lemma C.1. Let α be a point of S_Ω different from α_0. Then the interval $[\alpha_0 \times 0, \alpha \times 0]$ of L has the order type of $[0,1]$.

Proof. Note that the proof is trivial if α is the immediate successor of α_0 in S_Ω.

Suppose the lemma holds for all $\alpha < \beta$. We show it holds for β. If β has an immediate predecessor α_1, the proof is easy. The interval $[\alpha_0 \times 0, \alpha_1 \times 0]$ of L has the order type of $[0,1]$ by hypothesis. The interval $[\alpha_1 \times 0, \beta \times 0]$ of L equals $(\alpha_1 \times [0,1)) \cup \{\beta \times 0\}$, so it has the order type of $[0,1]$, and also of $[1,2]$. Their union has the order type of $[0,1] \cup [1,2] = [0,2]$, which of course has the order type of $[0,1]$.

If β has no immediate predecessor, there is an increasing sequence $\alpha_1, \alpha_2, \ldots$ of points of S whose supremum is β. Assume $\alpha_1 > \alpha_0$ for convenience. We show that for each i the interval $[\alpha_i \times 0, \alpha_{i+1} \times 0]$ of L has the order type of $[0,1]$. The interval $[\alpha_0 \times 0, \alpha_{i+1} \times 0]$ has the order type of $[0,1]$ by hypothesis; if $\alpha_i \times 0$ corresponds to the real number c of $[0,1]$ under the order-preserving bijection, then $[\alpha_i \times 0, \alpha_{i+1} \times 0]$ has the order type of $[c,1]$, which of course has the order type of $[0,1]$.

Finally, we note that the interval

$$J = [\alpha_0 \times 0, \beta \times 0)$$

of L can be written as the union

$$[\alpha_0 \times 0, \alpha_1 \times 0] \cup [\alpha_1 \times 0, \alpha_2 \times 0] \cup \ldots \cup [\alpha_i \times 0, \alpha_{i+1} \times 0] \cup \ldots$$

of intervals of L. There is an order-preserving correspondence of this union with the union

$$[0,1] \cup [1,2] \cup \ldots \cup [i, i+1] \cup \ldots$$

of intervals of $\mathbb{R}$. The latter union equals $[0,+\infty)$, which has the order type of $[0,1)$. When we adjoin the point $\beta \times 0$ to J, we obtain a set with the order type of $[0,1]$. $\square$

<u>Definition</u>. Let L' be the subspace $L - \{\alpha_0 \times 0\}$ of L; it is called the <u>Long Line</u>.

<u>Theorem C.2</u>. The long line is a path-connected linear continuum, every point of which has a neighborhood homeomorphic to an open interval of $\mathbb{R}$. It is not metrizable.

<u>Proof</u>. Let x be a point of L with $x \neq \alpha_0 \times 0$. Choose an element α of S_Ω so that $x < \alpha \times 0$. Then x lies in the open interval $(\alpha_0 \times 0, \alpha \times 0)$ of L, which has the order type of the open interval $(0,1)$ of $\mathbb{R}$.

The fact that L' is a linear continuum follows from Ex. 6 of §24. The result of the preceding paragraph shows that L' is the union of the open intervals $(\alpha_0 \times 0, \alpha \times 0)$ of L, each of which is path connected; since they have the point $\alpha_0 \times \frac{1}{2}$ in common, L' is path connected.

Now let α be the immediate successor of α_0 in S_Ω. We show that the ray $R = [\alpha \times 0, +\infty)$ of L' is limit point compact but not compact. It follows that R is not metrizable, so neither is L'.

The fact that R is not compact follows from the fact that the covering of L' by the open sets $[\alpha \times 0, \beta \times 0)$ with $\beta > \alpha$ has no finite (or even countable) subcovering. To show R is limit point compact, it suffices to show that every countably infinite set S in R has a limit point. And this is easy: The set of first coordinates of points of S has an upper bound in S_Ω. If β is the immediate successor of this upper bound, then S is a subset of the interval $[\alpha \times 0, \beta \times 0]$ of L'. Since L' is a linear continuum, this interval is compact; therefore S has a limit point. $\square$