We follow the pattern outlined in Exercises 2-7 on pp. 72-73 of the text.

Theorem B.1. Let J and E be well-ordered sets; let $h: J \to E$. Then the following are equivalent:

(i) h is order preserving and $h(J)$ equals E or a section of E.

(ii) $h(\alpha) = \text{smallest } [E - h(S_\alpha)]$ for each α.

Proof. Suppose (i) holds. Let β be an arbitrary element of J; let

$$e_0 = \text{smallest } [E - h(S_\beta)],$$

and suppose that $h(\beta) \neq e_0$.

It cannot be true that $h(\beta) < e_0$, for that would imply that $h(\beta)$ lies in $h(S_\beta)$, so that $h(\beta) = h(\alpha)$ for some $\alpha < \beta$; this in turn would contradict the fact that h is injective. Hence $h(\beta) > e_0$. Thus $h(J)$ contains an element greater than e_0.

Now we show that $h(J)$ does not contain e_0. Since $h(\beta) > e_0$ and h is order preserving, then for all $\alpha \geq \beta$, we have $h(\alpha) > e_0$. On the other hand, if $\alpha < \beta$, then $h(\alpha)$ belongs to $h(S_\beta)$, so that $h(\alpha) \neq e_0$ by definition of e_0.

Thus $h(J)$ contains an element greater than e_0, but does not contain e_0. This contradicts the fact that $h(J)$ equals E or a section of E.

Suppose (ii) holds. We first show that h is injective; this follows from the fact that if $\alpha < \beta$, then $h(\alpha)$ lies in $h(S_\beta)$, while by (ii), $h(\beta)$ does not. We then show that h is order preserving: Suppose $\alpha < \beta$. The set $h(S_\alpha)$ does not contain $h(\beta)$, since the statement "$h(\beta) = h(\gamma)$ for some $\gamma < \alpha$" would contradict the fact that h is injective. Since $h(\alpha)$ is the smallest element not in $h(S_\alpha)$, we have $h(\alpha) \leq h(\beta)$; equality cannot hold because h is injective.

If $h(J) = E$, the proof is complete. Suppose that $h(J) \neq E$; let e be the smallest element not in $h(J)$. Then $h(J)$ contains every element less than e. And $h(J)$ cannot contain any element greater than e; for if $h(\beta) > e$, then the fact that $h(\beta)$ is the smallest element not in $h(S_\beta)$ would imply that e belongs to $h(S_\beta)$ and hence to $h(J)$. We conclude that $h(J) = S_e$. □

Corollary B.2. Let J and E be well-ordered sets. There is at most one map $h: J \to E$ that is order preserving and whose image is E or a section of E.

Corollary B.3. If J is a well-ordered set, no section of J has the order type of J; nor can two different sections of J have the same order type.

Proof. If S_α is a section of J, then inclusion $i: S_\alpha \to J$ satisfies the conditions specified for the map h of the preceding corollary. Hence there is no surjective order-preserving map $h: S_\alpha \to J$. Similarly, if $\alpha < \beta$, then inclusion $i: S_\alpha \to S_\beta$ satisfies these same conditions, so there is no surjective order-preserving map $h: S_\alpha \to S_\beta$. □

Theorem B.4. Let J and E be well-ordered sets. If there is an order-preserving map $k: J \to E$, then there is an order-preserving map $h: J \to E$ whose image is E or a section of E.

Proof. Choose e_0 in E. By the principle of recursive definition, we may define a function $h: J \to E$ by setting

$$(*) \qquad h(\alpha) = \text{smallest}[E - h(S_\alpha)]$$

whenever $E - h(S_\alpha)$ is nonempty, and $h(\alpha) = e_0$ otherwise.

Now, given β, consider the following conditions:

(i) $h(\alpha) \leq k(\alpha)$ for all $\alpha < \beta$.

(ii) $E - h(S_\beta)$ is not empty.

(iii) $h(\beta) \leq k(\beta)$.

We show that (i) implies (ii) and (iii). Given (i), we have the inequalities $h(\alpha) \leq k(\alpha) < k(\beta)$ for $\alpha < \beta$, which imply that $k(\beta)$ does not belong to $h(S_\beta)$. Thus (ii) holds. It then follows from the definition of h that, since $h(\beta)$ is the smallest element of E not in $h(S_\beta)$, we have $h(\beta) \leq k(\beta)$.

The fact that (i) implies (iii) shows, by induction, that $h(\alpha) \leq k(\alpha)$ for all α. The fact that (i) implies (ii) then shows that h satisifes (*) for all α. We then apply Theorem B.1. □

Theorem B.5 (Comparability theorem). Let A and B be well-ordered sets. Exactly one of the following conditions holds:

(i) A has the order type of B.

(ii) A has the order type of a section of B.

(iii) B has the order type of a section of A.

Proof. Assume without loss of generality that A and B are disjoint. Order the set $C = A \cup B$ by using the order relations on A and on B, and by declaring that $a < b$ for a in A and b in B. It is easy to see that C is well-ordered.

Let b_0 be the smallest element of B. Then A equals the section of C by b_0. Inclusion $i : B \to C$ is order preserving; it follows from the preceding theorem that there is an order-preserving map $h : B \to C$ whose image is C or a section of C. If $h(B)$ equals the section of C by an element of A, then B has the order type of a section of A. If $h(B)$ equals the section of C by b_0, then B has the order type of A. And if $h(B)$ equals the section of C by an element $b > b_0$ of B, or if $h(B)$ equals all of C, then A has the order type of a section of B.

The preceding corollary implies that only one of the conditions (i)-(iii) can hold. □

Lemma B.6. Let X be a set; let $\mathcal{A}$ be the collection of all pairs $(A,<)$, where A is a subset of X and $<$ is a well-ordering of A. Define

$$(A,<) \prec (A',<')$$

if $(A,<)$ equals a section of $(A',<')$. Then $\prec$ is a strict partial order on $\mathcal{A}$. If $\mathcal{B}$ is a simply ordered subcollection of $\mathcal{A}$, let C equal the union of the sets B, for all $(B,<)$ in $\mathcal{B}$; and let $<_C$ equal the union of the relations $<$, for all $(B,<)$ in $\mathcal{B}$. Then $(C,<_C)$ is an upper bound for $\mathcal{B}$ in $\mathcal{A}$.

Proof. We check the conditions for a strict partial order. Nonreflexivity is immediate, for A cannot equal a section of itself. Transitivity is also immediate, since if A_1 is a section of A_2 and A_2 is a section of A_3, then A_1 is a section of A_3.

Now consider the set C. Given two distinct elements b_0 and b_1 of C, there is an element $(B,<)$ of $\mathcal{B}$ such that B contains both of them (because $\mathcal{B}$ is simply ordered by $\prec$). One of these elements is less than the other under $<$, and which relation holds is independent of the choice of $(B,<)$, again because $\mathcal{B}$ is simply ordered. Hence one is less than the other under $<_C$.

Since the relation $b<b$ cannot hold in B, for any $(B,<)$ in $\mathcal{B}$, we cannot have $b<_C b$.

Finally, suppose b_0, b_1, and b_2 are elements of C such that

$$b_0 <_C b_1 <_C b_2 .$$

Because $\mathcal{B}$ is simply ordered, there is an element $(B,<)$ in $\mathcal{B}$ such that b_0, b_1, and b_2 all belong to B and the relations $b_0 < b_1$ and $b_1 < b_2$ hold in B. Then the relation $b_0 < b_2$ holds in B, so that we have $b_0 <_C b_2$.

Therefore C is simply ordered; we show C is well-ordered. Let D be an arbitrary nonempty subset of C. Then D intersects some set B_0, where $(B_0, <_0)$ belongs to $\mathcal{B}$. Let us take the smallest element d of $D\cap B_0$ in the well-ordered set $(B_0,<_0)$. This element is independent of the choice of B_0. For if $(B_1, <_1)$ is another element of $\mathcal{B}$ such that D intersects B_1, then one of $(B_0, <_0)$ and $(B_1, <_1)$ equals a <u>section</u> of the other, so that the smallest elements of $D\cap B_0$ and $D\cap B_1$ are the same. A similar argument shows that d is the smallest element of C.

Finally, we must show that $(C, <_C)$ is an upper bound for $\mathcal{B}$; that is, given an element $(B,<)$ of $\mathcal{B}$, either $(B,<)$ equals $(C, <_C)$ or it equals a section of $(C,<_C)$. We know that $B\subset C$ and that $<$ is contained in $<_C$. Suppose that equality does not hold. Let c be the smallest element of C that is not in B. Then B contains the section of C by c. We show that B contains no element c_0 of C that is greater than c; this implies that B <u>equals</u> the section of C by c.

So suppose B contains $c_0 > c$. As before, there is an element $(B_0, <_0)$ of $\mathcal{B}$ such that B_0 contains both c_0 and c. B_0 cannot be a section of B because B does not contain c. And B cannot be a section of B_0 because B contains c_0 but not the smaller element c. Thus we reach a contradiction to the fact that $\mathcal{B}$ is simply ordered. $\square$

Theorem B.7. The maximum principle is equivalent to the well-ordering theorem.

Proof. We have sketched in the text (p.70) how one can use the principle of recursive definition to show that the well-ordering theorem implies the maximum principle.

The preceding lemma provides a proof of the reverse implication. Given a set X, one proceeds as in the lemma. The maximum principle gives one a maximal subcollection $\mathcal{B}$ that is simply ordered by $\prec$. Its upper bound C must equal all of X, for if x were an element of X not in C, one could form a larger well-ordered set D by adjoining x to C and declaring x to be larger than every element of C. Then C would equal the section of D by x. Adjoining D to the collection $\mathcal{B}$ would give us a simply ordered subcollection of $\mathcal{A}$ that properly contains $\mathcal{B}$, contradicting maximality. □

Theorem B.8. The choice axiom is equivalent to the well-ordering theorem.

Proof. It is immediate that the well-ordering theorem implies the choice axiom. We prove the converse.

Given X, let c be a choice function for the nonempty subsets of X. If T is a subset of X and $<$ is a relation on T, we say that $(T,<)$ is a tower in X if $<$ is a well-ordering of T and if for each x in T,

$$x = c(X - S_x(T)),$$

where $S_x(T)$ is the section of T by x.

Step 1. Given two towers $(T_1,<_1)$ and $(T_2,<_2)$ in X, either they are equal or one equals a section of the other.

Switching indices if necessary, the comparability theorem tells us there is an order-preserving map

$$h : T_1 \to T_2$$

whose image is either T_2 or a section of T_2. Theorem B.1 tells us that h

must be given by the formula

$$(*) \qquad h(x) = \text{smallest}[T_2 - h(S_x(T_1))].$$

This in turn implies that $h(x) = x$ for all x in T_1, as we now show:

We proceed by transfinite induction. Suppose that y is in T_1 and that $h(x) = x$ for all $x < y$. We show $h(y) = y$.

Consider the restricted function $h : S_y(T_1) \to T_2$. Because (*) holds, the image must be a section of T_2. (It cannot equal T_2, because it does not contain $h(y)$.) This section is of course the section by the element

$$\text{smallest } [T_2 - h(S_y(T_1))],$$

which by (*) is just $h(y)$. Thus

$$h(S_y(T_1)) = S_{h(y)}(T_2).$$

It follows that

$$\begin{aligned} h(y) &= c(X - S_{h(y)}(T_2)) \text{ by definition of a tower,} \\ &= c(X - h(S_y(T_1)) \text{ as just noted,} \\ &= c(X - S_y(T_1)) \text{ because } h(x) = x \text{ for } x < y, \\ &= y \text{ by definition of a tower.} \end{aligned}$$

Thus $h(x) = x$ for all x in T_1. It follows that $h(T_1) = T_1$, so that T_1 equals either T_2 or a section of T_2.

<u>Step 2</u>. Let $(T_i, <_i)$ be the collection of all towers in X. Let T be the union of all the sets T_i and let $<$ be the union of all the relations $<_i$. We show that $(T, <)$ is a tower in X.

We showed in Step 1 that the collection of all towers in X is simply ordered by the relation $\prec$ of Lemma B.6. It follows from this lemma that $(T, <)$ is a well-ordered set. We show that it is actually a tower.

This is in fact easy. Given x in T, we must show that

$$x = c(X - S_x(T)).$$

Now there is a tower $(T_1, <_1)$ in X such that T_1 contains x. By Lemma B.6, T_1 equals T or a section of T. Therefore, $S_x(T_1) = S_x(T)$. Because T_1 is a tower,

$$x = c(X - S_x(T_1));$$

our desired result follows.

Step 3. We show that $T = X$. If T is not all of X, we can set

$$y = c(X - T),$$

and make the set $T \cup \{y\}$ into a well-ordered set by declaring $y > x$ for every x in T. Then not only is this set well-ordered, it is also a tower in X. This contradicts the fact that T is obtained by taking the union of all towers in X. □

EXERCISES

1. Suppose we alter the statement of Lemma B.6 by declaring that $(A, <) \prec (A', <')$ if A is contained in A' and $<$ is contained in $<'$. Show that the resulting set C is simply ordered. Give an example to show that it need not be well-ordered.

2. Let $\mathcal{C}$ be a collection of sets. Let us define two sets to be equivalent if there is a bijection between them; the equivalence classes are called cardinal numbers. Let us denote the equivalence class of the set A by $c(A)$; and let us define $c(A) < c(B)$ if there is an injection $i : A \to B$ but no injection of B into A. Show that this is a well-defined relation, and that this collection of cardinal numbers is well-ordered by this relation.

The cardinal number of the positive integers is commonly denoted $\aleph_0$. (Read "aleph naught.") The next cardinal number after this one is denoted (surprise!) $\aleph_1$. The cardinal number of the reals is denoted $\underline{c}$ ("the cardinality of the continuum"). The continuum hypothesis is the statement that $\aleph_1 = c$.

[It is tempting to try to construct the collection of all cardinal numbers by beginning with the collection of all sets and introducing the above equivalence relation. The problem is that the collection of all sets is a contradictory notion. See Exercise 6 of §9. Logicians have formulated a way around this difficulty, so that they can consider arbitrarily large cardinal numbers.]