18.786 Problem Set 4 (due Thursday Mar 4 in class)

1. Let $| \ |_1$ and $| \ |_2$ be two norms on a field K. Show that they are equivalent (i.e. $|x|_1 < 1 \iff |x|_2 < 1$ for all $x \in K$) if and only if there is a positive real number s such that $|x|_1 = |x|_2^s$ for all $x \in K$. Also show that they are equivalent iff they induce the same topology on K.

2. For which $s > 0$ is $| \ |_\infty^s$ on $\mathbb{Q}$ an absolute value? Prove your answer.

3. Let $f(X) = \sum_{n=0}^{\infty} a_n X^n$ be a power series in $\mathbb{Q}_p[[X]]$, and define $\rho = \frac{1}{\limsup \sqrt[n]{|a_n|}}$ (here $| \ | = | \ |_p$ is the p-adic norm). Suppose that $0 < \rho < \infty$. Show that

   - If $\lim_{n\to\infty} |a_n|\rho^n$ exists and equals 0, then $f(x)$ converges for $x \in \mathbb{Q}_p$ iff $|x| \le \rho$.
   - If $|a_n|\rho^n$ does not tend to 0 as $n \to \infty$, then $f(x)$ converges for $x \in \mathbb{Q}_p$ iff $|x| < \rho$. Find the domain of convergence for the exponential and logarithmic series:

   $$\exp(X) = \sum_{n=0}^{\infty} \frac{X^n}{n!}, \quad \log(1+X) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{X^n}{n}$$

4. Let K be a field complete for a nonarchimedean valuation, and extend the valuation of K (uniquely) to the algebraic closure. Now suppose α, β are elements of $\overline{K}$, such that α is separable over $K(\beta)$, and such that for all conjugates $\alpha_i \neq \alpha$ of α over $K(\beta)$, we have

   $$|\beta - \alpha| < |\alpha_i - \alpha|.$$

   Then show that $\alpha \in K(\beta)$.

5. Let K be a field complete for the nonachimedean exponential valuation v. Let $f(X) = a_0 + a_1 X + \cdots + a_n X^n \in K[X]$ be a polynomial with $a_0 a_n \neq 0$. Now to this polynomial, associate a finite polygonal chain, which is the lower convex hull of the points $(i, v(a_i))$ in the plane. This is called the *Newton polygon* of f. Show that to every line segment of this polygon of slope $-m$, from say $(r, v(a_r))$ to $(s, v(a_s))$, correspond exactly $s - r$ roots of f of valuation m (in the splitting field of f).

6. Use gp/Pari to compute the class numbers of quadratic fields $\mathbb{Q}(\sqrt{d})$ for $|d| < 1000$, and report on any patterns noticed. In particular, pay attention to the factors of 2.

7. Let $f(x) = 8x^3 - 6x - 1$: this is the minimal polynomial of $\cos(\pi/9)$, which is used to show that we cannot trisect a general angle using ruler and compasses. Use Hensel's lemma (or modify Newton's method p-adically) to find a solution in $\mathbb{Q}_{17}$ to precision $O(17^{20})$ [write a loop, do not use polrootspadic()].