Artin and Brauer Reciprocity, Part I

Let $F/\mathbb{Q}$ be a global field, so that we have an exact sequence

$$1 \to F^\times \hookrightarrow \mathbb{A}_F^\times \to C_F \to 1,$$

where $C_F := \mathbb{A}_F^\times/F^\times$. Last time, we almost showed that for a cyclic degree-n extension E/F, we have $\chi(C_E) = n$; it remains to prove Lemma 20.14.

LEMMA 21.1. *Let L/K be an extension of infinite fields, A be a K-algebra, and M and N be two A-modules that are finite dimensional over K with M projective over A. If $M \otimes_K L \simeq N \otimes_K L$ as $A \otimes_K L$-modules, then $M \simeq N$.*

PROOF. First note that there is an isomorphism

(21.1) $$\operatorname{Hom}_A(M, N) \otimes_K L \xrightarrow{\sim} \operatorname{Hom}_{A \otimes_K L}(M \otimes_K L, N \otimes_K L).$$

Indeed, because M is a finitely generated projective module, it is a summand of a finite-rank free module, which reduces us to the case $M = A$ (since the Hom functor commutes with direct sums). Then both sides of (21.1) reduce to $N \otimes_K L$. (In fact, a more basic identity holds in greater generality: $\operatorname{Hom}_A(M, N \otimes_A P) = \operatorname{Hom}_A(M, N) \otimes_A P$ for an arbitrary algebra A as long as P is flat and M is finitely presented, i.e., $A^{n_1} \to A^{n_2} \to M \to 0$ is exact for some $n_1, n_2 \in \mathbb{Z}$).

Observe that both $M \otimes_K L$ and $N \otimes_K L$ have the same dimension over L, hence M and N have the same dimension d over K. Let $V := \operatorname{Hom}_A(M, N)$ and $W := \operatorname{Hom}_A(\Lambda^d M, \Lambda^d N)$, where $\Lambda^d(-)$ denotes the dth exterior power. Both of these are finite-dimensional K-vector spaces; in particular, V is d^2-dimensional, and W is 1-dimensional, i.e., isomorphic to K. Functoriality of Λ^d gives the determinant map $\det \colon V \to W$, which is a polynomial map of degree d with coefficients in K, in the sense that after choosing bases of V and W, it is given by a degree-d polynomial in coordinates (as it is computed as the determinant of a $d \times d$ matrix in V). We'd like to show that the map $\det$ is non-zero, as a point $\varphi \in V$ with $\det \varphi \neq 0$ is the same as an A-module isomorphism $M \simeq N$. Since K is infinite (and $\det$ is polynomial), it suffices to check this after extending scalars to L, which by (21.1) gives a determinant map

$$\operatorname{Hom}_{A \otimes_K L}(M \otimes_K L, N \otimes_K L) \to \operatorname{Hom}_{A \otimes_K L}(\Lambda^d(M \otimes_K L), \Lambda^d(N \otimes_K L)) \simeq L.$$

Since the left-hand side contains an isomorphism $M \otimes_K L \simeq N \otimes_K L$, this map must be non-zero, as desired. $\square$

PROOF (OF LEMMA 20.14). The issue here is that Λ_1 and Λ_2 might not be commensurate (i.e., each is contained in the other up to a finite index and multiplication). However, we claim that $\Lambda_1 \otimes \mathbb{Q} \simeq \Lambda_2 \otimes \mathbb{Q}$ as $\mathbb{Q}[G]$-modules. Indeed, by Lemma 21.1, it suffices to show that $\Lambda_1 \otimes \mathbb{R} \simeq \Lambda_2 \otimes \mathbb{R}$ as $\mathbb{R}[G]$-modules, which is clear as $\Lambda_i \otimes \mathbb{R} \simeq V$ for $i = 1, 2$. Thus, taking the image of Λ_2 under this isomorphism, and

tensoring with $\mathbb{R}$ to obtain inclusion in V, there exists a G-stable lattice $\Lambda_3 \subseteq V$ that is isomorphic to Λ_2 as a $\mathbb{Z}[G]$-modules and commensurate with Λ_1. Thus, $N\Lambda_3 \subseteq \Lambda_1 \subseteq \frac{1}{N}\Lambda_3$ for some sufficiently large N, hence $\chi(\Lambda_1) = \chi(\Lambda_3) = \chi(\Lambda_2)$ as all subquotients are finite, as desired. $\qquad\square$

We now turn to a discussion of local Kronecker–Weber theory.

THEOREM 21.2. *For any prime p,*

$$\mathbb{Q}_p^{\mathrm{ab}} = \left[\bigcup_{n \geq 0} \mathbb{Q}_p(\zeta_{p^n}) \right] \cdot \mathbb{Q}_p^{\mathrm{unr}},$$

where this is the compositum with the non-completed maximal unramified extension of $\mathbb{Q}_p$.

PROOF. Recall that the extensions $\mathbb{Q}_p(\zeta_{p^n})/\mathbb{Q}_p$ are totally ramified, with Galois groups $(\mathbb{Z}/p^n\mathbb{Z})^\times$. Thus, $K \subseteq \mathbb{Q}_p^{\mathrm{ab}}$, where we have denoted the right-hand side by K, and this gives a map

$$\mathbb{Z}_p^\times \times \widehat{\mathbb{Z}} = \widehat{\mathbb{Q}_p^\times} = \mathrm{Gal}(\mathbb{Q}_p^{\mathrm{ab}}/\mathbb{Q}_p) \twoheadrightarrow \mathrm{Gal}(K/\mathbb{Q}_p) = \left[\varprojlim_n (\mathbb{Z}/p^n\mathbb{Z})^\times \right] \times \widehat{\mathbb{Z}},$$

where the left-most equalities are by LCFT, followed by choice of a uniformizer of $\mathbb{Q}_p$; the map is surjective by Galois theory. Since both sides are isomorphic as abstract groups, the following lemma shows that this map is an isomorphism. $\quad\square$

LEMMA 21.3. *Let G be a profinite group, such that for all $n > 0$, the number of open subgroups of index n in G is finite (i.e., G is "topologically finitely generated"). Then every continuous homomorphism $\varphi\colon G \twoheadrightarrow G$ is an isomorphism.*

PROOF. If $H \subseteq G$ is a subgroup of index at most n, then $\varphi^{-1}(H) \subseteq G$ is also a subgroup of index at most n. Thus, we have a map

$$\{H \subseteq G : [G : H] \leq n\} \xrightarrow{\varphi^{-1}} \{H \subseteq G : [G : H] \leq n\},$$

which is injective as φ is surjective. By hypothesis, this set is finite, hence this map is bijective. Since

$$\mathrm{Im}(\varphi^{-1}) = \{H \subseteq G : [G : H] \leq n, \mathrm{Ker}(\varphi) \subseteq H\},$$

it follows that $\mathrm{Ker}(\varphi)$ is contained in every finite-index subgroup of G, hence $\mathrm{Ker}(\varphi)$ is trivial and φ is an isomorphism. $\qquad\square$

We now ask: what is the automorphism of $\mathbb{Z}_p^\times \times \widehat{\mathbb{Z}}$ in the proof of Theorem 21.2? The following theorem of "explicit CFT" answers this question, but the proof is involved and not at all obvious (see [**Dwo58**]). An answer to the analogous question for global fields is not known in general, aside from the cases of $\mathbb{Q}$ and imaginary number fields.

THEOREM 21.4 (Dwork, Lubin–Tate). *(1) The element $p \in \mathbb{Q}_p^\times$ acts trivially on $\bigcup_n \mathbb{Q}_p(\zeta_{p^n})$ and acts as the Frobenius element on $\mathbb{Q}_p^{\mathrm{unr}}$.*
(2) An element $x \in \mathbb{Z}_p^\times$ acts trivially on $\mathbb{Q}_p^{\mathrm{unr}}$ and acts by x^{-1} on $\bigcup_n \mathbb{Q}_p(\zeta_{p^n})$, i.e., $\theta_p(x) \cdot \zeta_{p^n} = \zeta_{p^n}^{(x^{-1} \bmod p^n)}$, where

$$\theta_p \colon \mathbb{Q}_p^\times \to \mathrm{Gal}\Big(\bigcup_n \mathbb{Q}_p(\zeta_{p^n}) \cdot \mathbb{Q}_p^{\mathrm{unr}} \Big/ \mathbb{Q}_p \Big)$$

is the homomorphism provided by LCFT.

There are two reciprocity laws which we'd now like to introduce: *Artin reciprocity*, and *Brauer reciprocity*. We'll begin with the former. Let E/F be an abelian G-Galois extension of global fields. Recall that we expect to have

$$F^\times \backslash \mathbb{A}_F^\times / \mathrm{N}(\mathbb{A}_E^\times) = C_F / \mathrm{N}(C_E) \xrightarrow{\sim} G.$$

We'd like to construct this map.

CLAIM 21.5. LCFT *gives us a map* $\theta \colon \mathbb{A}_F^\times \to G$ *with* $\theta(\mathrm{N}(\mathbb{A}_E^\times)) = 1$.

PROOF. Let $x \in \mathbb{A}_F^\times$. For each $v \in M_F$, we have an element $x_v \in F_v^\times$, and LCFT then gives a map

$$\theta_v \colon F_v^\times \to \mathrm{Gal}(E_w/F_v) \subseteq \mathrm{Gal}(E/F) = G$$

for a place $w \mid v$ of E, where the former is the decomposition group of E/F at v. This embedding is induced by the embedding $E \subseteq E_w$. Recall that when E/F is abelian, $\mathrm{Gal}(E_w/F_v)$ is independent of the choice of w.

We now claim that the product

$$\theta(x) := \prod_{v \in M_F} \theta_v(x_v)$$

makes sense, that is, $\theta_v(x_v) = 1$ for all but finitely many v. Indeed, for almost all v, E_w/F_v is unramified and $x_v \in \mathcal{O}_{F_v}^\times$, implying that $\theta_v(x_v) = 1$ (since by LCFT, the map θ_v kills $\mathcal{O}_{F_v}^\times$ and sends a uniformizer of F_v to the Frobenius element of G).

Since $\theta_v(\mathrm{N}(E_w^\times)) = 1$ for all $v \in M_F$, we have $\theta(\mathrm{N}(\mathbb{A}_E^\times)) = 1$, as desired.   □

THEOREM 21.6 (Artin Reciprocity). *We have* $\theta(F^\times) = 1$, *hence* θ *gives a map* $C_F / \mathrm{N}(C_E) \to G$.

EXAMPLE 21.7. If $E/\mathbb{Q}$ is a quadratic extension, this reduces to quadratic reciprocity. Indeed, the local Artin maps are simply given by Hilbert symbols, and from here we proved the implication.

We will proof this concurrently with Brauer reciprocity. Let E/F be a finite G-extension of global fields. We have

$$H^2(G, \mathbb{A}_E^\times) = \bigoplus_{v \in M_F} \mathrm{Br}(F_v) = \bigoplus_{v \in M_F \backslash M_F^\infty} \mathbb{Q}/\mathbb{Z} \times \bigoplus_{v \in M_F^\infty} \tfrac{1}{2}\mathbb{Z}/\mathbb{Z},$$

by Claim 20.10 and since $\mathrm{Br}(\mathbb{R}) = \mathbb{Z}/2\mathbb{Z}$. Define the *invariant map* $\iota \colon H^2(G, \mathbb{A}_E^\times) \to \mathbb{Q}/\mathbb{Z}$ via

$$\bigoplus_{v \in M_F / M_F^\infty} \mathbb{Q}/\mathbb{Z} \xrightarrow{(x_v)_v \mapsto \sum_v x_v} \mathbb{Q}/\mathbb{Z},$$

i.e., summing over all local factors (and ignoring all infinite ones).

THEOREM 21.8 (Brauer Reciprocity). *The composition*

$$\mathrm{Br}(F/E) \to H^2(G, \mathbb{A}_E^\times) \xrightarrow{\iota} \mathbb{Q}/\mathbb{Z}$$

is zero for all $E/\mathbb{Q}$.

Note that E is essentially irrelevant here; this theorem is really about $\mathrm{Br}(F)$. The first map is induced by the diagonal embedding $E \hookrightarrow \mathbb{A}_E^\times$, and the cohomological interpretation of the Brauer group then shows that every division algebra over F is a matrix algebra except at finitely many places, which is otherwise not an obvious statement.

EXAMPLE 21.9. Any $a, b \in \mathbb{Q}^\times$ give a Hamiltonian algebra $H_{a,b}$ over $\mathbb{Q}$. At a finite prime p, the invariant is

$$\begin{cases} 0 & \text{if } H_{a,b} \otimes \mathbb{Q}_p \simeq \mathrm{M}_2(\mathbb{Q}_p), \\ 1/2 & \text{otherwise.} \end{cases} \quad \Longleftrightarrow \quad \begin{cases} 0 & \text{if } (a,b)_p = 1, \\ 1/2 & \text{if } (a,b)_p = -1. \end{cases}$$

Thus, asserting that the sum of all invariants is zero is again quadratic reciprocity.

CLAIM 21.10. *Artin reciprocity is valid for $\mathbb{Q}(\zeta_n)/\mathbb{Q}$, where ζ_n is a primitive nth root of unity.*

PROOF. We proceed via explicit calculation using Dwork's theorem. We may assume that $n = \ell^r$ is a prime power, because $\mathbb{Q}(\zeta_n)$ is the compositum of $\mathbb{Q}(\zeta_{\ell^r})$ over all prime-power factors ℓ^r of n, and $\mathrm{Gal}(\mathbb{Q}(\zeta_n)/\mathbb{Q})$ then splits as a product over $\mathrm{Gal}(\mathbb{Q}(\zeta_{\ell^r})/\mathbb{Q})$. We then have a composition

$$\mathbb{Q}^\times \hookrightarrow \mathbb{A}_{\mathbb{Q}}^\times \xrightarrow{\theta} \mathrm{Gal}(\mathbb{Q}(\zeta_{\ell^r})/\mathbb{Q}) = (\mathbb{Z}/\ell^r\mathbb{Z})^\times,$$

where $\theta = \prod_p \theta_p$ as before. We'd like to show that this map is trivial. To this end, it suffices to show that $\theta(p) = 1$ for all primes p and $\theta(-1) = 1$. Suppose $p \neq \ell$; the case $p = \ell$ will be covered in the next lecture. Then

$$\begin{cases} \theta_p(p) = p, \\ \theta_\ell(p) = p^{-1}, \\ \theta_q(p) = 1, \\ \theta_\infty(p) = 1. \end{cases}$$

Indeed, recall that

$$\theta_p \colon \mathbb{Q}_p^\times \to \mathrm{Gal}(\mathbb{Q}_p(\zeta_{\ell^r})/\mathbb{Q}_p)$$
$$p \mapsto \mathrm{Frob}_p,$$

and $\theta(\mathbb{Z}_p^\times) = 1$. But Frob_p corresponds to $p \in (\mathbb{Z}/\ell^r\mathbb{Z})^\times$. For the second case, $\theta_\ell(p)$ acts as $p^{-1} \in (\mathbb{Z}/\ell^r\mathbb{Z})^\times$ by Dwork's theorem, and for the third case, in which $q \neq p, \ell$, we have $p \in (\mathbb{Z}/q\mathbb{Z})^\times$ and the extension $\mathbb{Q}_q(\zeta_{\ell^r})/\mathbb{Q}_q$ is unramified. Finally, θ_∞ corresponds to taking sign, as it is a map $\mathbb{R}^\times \to \mathrm{Gal}(\mathbb{C}/\mathbb{R})$ which contracts the connected components of $\mathbb{R}^\times$, giving a map from $\mathbb{Z}/2\mathbb{Z}$ sending 1 to complex conjugation. $\square$

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