18.786 PROBLEM SET 7

(1) Let G be a finite group and let H be a subgroup.
  (a) Consider $\mathbb{Z}[G]$ as a left $\mathbb{Z}[H]$-module. Show that it is a finite rank free module over $\mathbb{Z}[H]$.
  (b) For a finitely generated projective left $\mathbb{Z}[H]$-module P, show that its dual $P^\vee := \mathrm{Hom}_H(P, \mathbb{Z}[H])$ is naturally a *right* $\mathbb{Z}[H]$-module, and as such is finitely generated and projective. Then show that the dual to $\mathbb{Z}[G]$ is canonically isomorphic to $\mathbb{Z}[G]$, thought of now as a right $\mathbb{Z}[H]$-module (by letting H act on G on the right).
  (c) Show that for any complex X of H-modules, there is a canonical quasi-isomorphism:
$$(\mathbb{Z}[G] \underset{\mathbb{Z}[H]}{\otimes} X)^{hG} \simeq X^{hH}.$$
  (d) Show that for any complex X of H-modules, there is a canonical quasi-isomorphism:
$$(\mathbb{Z}[G] \underset{\mathbb{Z}[H]}{\otimes} X)^{tG} \simeq X^{tH}.$$
  (e) Show that for any complex A of abelian groups, $A[G] := \mathbb{Z}[G] \otimes_{\mathbb{Z}} A$ has $(A[G])^{tG} = 0$. In particular, show that $\mathbb{Z}[G]^{tG} = 0$.

(2) Let G and H be as above. Let X be a complex of G-modules.

Recall from class that we have a *restriction* map $X^{hG} \to X^{hH}$ and an *inflation* map $X^{hH} \to X^{hG}$, and that the composition $X^{hG} \to X^{hH} \to X^{hG}$ is homotopic to multiplication by the index $[G : H]$.[1]
  (a) Show that there are canonical maps $X_{hH} \to X_{hG}$ and $X_{hG} \to X_{hH}$, such that the composed endomorphism of X_{hG} is again homotopic to multiplication by $[G : H]$.
  (b) Do the same for Tate cohomology.
  (c) Show that multiplication by $|G|$ is nullhomotopic on X^{tG}. Deduce that $\widehat{H}^i(G, X) := H^i(X^{tG})$ is a $\mathbb{Z}/|G|$-module.

(3) Let L/K be a Galois extension of fields with Galois group G. In this problem, we will show that $H^1(G, L^\times) = 0$, a generalization of Hilbert's Theorem 90 due to Noether (and often just referred to as Hilbert's Theorem 90).

Suppose that $\varphi : G \to L^\times$ is a group 1-cocycle, i.e., we have the identity:

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[1] Briefly, the construction went by identifying X^{hH} with $\underline{\mathrm{Hom}}_G^{der}(\mathbb{Z}[G/H], X)$ and then using the canonical G-equivariant maps $\mathbb{Z}[G/H] \to \mathbb{Z}$ and $\mathbb{Z} \to \mathbb{Z}[G/H]$. You should think that restriction is about regarding a G-invariant vector as an H-invariant one, and inflation is about averaging an H-invariant vector to a G-invariant vector via $x \mapsto \sum_{g \in G/H} g \cdot x$.

$$\varphi(gh) = \varphi(g) \cdot (g \cdot \varphi(h)).$$

(Here the first $\cdot$ is multiplication in L, and the second $\cdot$ is the action of g on L.)

We need to show that φ is coboundary, i.e., that there exists $x_0 \in L^\times$ with:

$$\varphi(g) = \frac{x_0}{g \cdot x_0}$$

for all $g \in G$.

(a) Remind me: why is this enough to deduce that $H^1(G, L^\times) = 0$?
(b) Define $T_g : L \to L$ by $T_g(x) = \varphi(g) \cdot (g \cdot x)$. Show that $T_{gh} = T_g \circ T_h$.
(c) Show that $\sum_{g \in G} T_g$ is a non-zero K-linear map $L \to L$.
(d) Deduce that φ is a coboundary.

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